REPORT RESUMES

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PRECISION IN RESEARCH ON COMPLEX LEARNING AND TEACHING--THE MATHEMATICAL FORMULATION OF EDUCATIONAL RESEARCH QUESTIONS.

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THE OBJECTIVE OF THIS RESEARCH WAS TO EXTEND, ELABORATE, AND IMPROVE THE SET-FUNCTION LANGUAGE (SFL). THE SFL IS A NEW SCIENTIFIC LANGUAGE FOR FORMULATING RESEARCH QUESTIONS ON MEANINGFUL LEARNING, FRAMED IN TERMS OF THE MATHEMATICAL NOTIONS OF SETS AND FUNCTIONS. IN THE FIRST PAPER, THE SFL IS DESCRIBED. ITS RELATION TO THE STIMULUS RESPONSE MEDIATION LANGUAGE (SR) IS DISCUSSED. A METHODOLOGY FOR ASSESSING WHAT IS LEARNED AND A REFORMULATION OF RESEARCH QUESTIONS, IN TERMS OF THE SFL AND THE RELATED ASSESSMENT METHODOLOGY, ARE PRESENTED. THE SECOND SECTION DEALS WITH EFFORTS TO REFINE AND EXTEND THE SFL. THE AUTHOR SHOWS HOW THE SFL AIDED IN THE DEVELOPMENT OF SOME RESEARCH, AND PROBLEMS CONCERNING RECEPTION AND DISCOVERY LEARNING, REVERSAL AND NONREVERSAL SHIFTS. PIAGETIAN CONSERVATION TASKS, AND SYMBOLIC AND CONCRETE LEARNING CAN BE REFORMULATED IN THE SFL. THE THIRD PAPER PROVIDES IN-DEPTH ANALYSES OF CERTAIN QUESTIONS RELATED TO MATH AND SCIENCE EDUCATION. THE NEXT SECTION CONTRASTS SFL AND SR FORMULATIONS OF SEVERAL MEANINGFUL LEARNING TASKS. THE FINAL PAPER GLEANS SOME OF THE HIGHLIGHTS OF BOTH THEORY AND EMPIRICAL RESEARCH BASED ON THE SFL. FUTURE DIRECTIONS FOR PSYCHOMATHEMATICS AND THE POSSIBLE RESULTS OF A MAJOR REFINEMENT OF THE SFL ARE DISCUSSED. THE AUTHOR WEIGHTS THE RESULTS, THE CONCLUSIONS, AND THE IMPLICATIONS OF THE PROJECT. (PS)

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THE BASIC UNIT IN MEANINGFUL LEARNING--

ASSOCIATION OR PRINCIPLE?
(A Set-Function Language)

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Theoretical development in educational psychology has been extremely slow. A major reason has been the lack of an explicit language with which to discuss research on meaningful learning and teaching. As McDonald (1964, 542) has put it "...occasional attempts to make a confusing conceptual formulation understandable are not sufficient. Conceptual clarity means (a) specification, stated in terms as nearly operational as possible, of the behavior involved in a task or method; (b) some delineation of the range of phenomena included and excluded; and (c) precise description of the appropriate tests."

Stating research objectives and defining variables in unambiguous terms is not sufficient. The teaching-learning process has all too frequently been studied in terms of traditionally defined categories. Much research has been of a fragmentary nature; similarities and essential differences have gone undetected. The variables chosen need to have general relevance, not be inextricably related to the question at hand. Without this characteristic, research findings, almost of necessity remain isolated.

To provide a substantive base for their research, educational psychologists have frequently resorted to the languages, paradigms, and theories of the mother science of psychology. Mediational elaborations of the S-R language, operant conditioning paradigms, and more general, but less well specified, cognitive theories have been popular.

Each approach has important limitations. From one point of view, parsimony suggests that the properties of overt S-R associations should also be attributed to mediational links. Yet, practice has shown that mediational interpretations become increasingly cumbersome and less precise as situations become more complex. Similar difficulties have plagued researchers who have used operant techniques to study meaningful verbal learning. The results simply are no where near as clear in complex human learning as they are in the "Skinner Box." It is increasingly recognized, for example, that knowledge of results is not directly analogous to feeding a pigeon and that, in any case, other factors, such as subject matter structure, are probably of greater importance in promoting efficient learning (e.g., Bruner, 1960; Gagné, 1961). A general limitation of cognitive theories is their relative imprecision. Typically, "cognitions" are either not clearly specified in observable terms or are only partially defined.[1]

A most important outcome of recent collaborative efforts between educational psychologists and subject matter specialists has been to focus attention on the close relationships between task and method variables. Subject matter educators and psychologists have increasingly come to realize that research on meaningful learning and teaching must, on the one hand, deal with observables (i.e. behavior) and, on the other hand, with subject matter structure. Little of scientific and practical value to education can be accomplished by talking about either unobservable mental processes or structure-free (i.e. rote) materials. There is need for a precise language, couched in observables, which also provides for the description of psychologically relevant subject matter characteristics.

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[1] These criticisms in no way deny the importance of existent psychological theories. The S-R theories have proven invaluable in dealing with simple learning while cognitive theories provide much needed structure and explanatory mechanisms for extremely complex phenomena.

A Set-Function Language (SFL) has been devised to meet this need. By disregarding certain of the subtleties involved in simpler forms of learning, while representing gross characteristics, the SFL makes it possible to deal with many questions relating to meaningful learning and teaching in a precise manner. This language not only provides a symbolic means of describing certain research problems, but makes it possible to consider structural and behavioral questions simultaneously.

There are several ways in which the SFL can be used to make research on teaching and learning more explicit. First, learning objectives can be well defined (Gagné, 1965). Differences in learning type can symbolically be represented. Second, distinctions can be made between presented information. It is possible, for example, to specify the difference between presenting a principle (i.e. rule) directly as opposed to presenting an instance of the principle. Third, the important question of "what is learned" can be pointedly discussed. The sort of stage theory of instuction and problem solving, to which Gagné (1962, 1964) alludes, ultimately will depend on the ability to assess learning type. After establishing the acquisition of specific knowledge, the question always remains as to whether the learning was rote or meaningful. Fourth, predictions can be stated in unequivocable terms which relate to the other distinctions made. Only by being able to discuss relationships between inputs, prior knowledge, and the criterion task can research, on meaningful learning and teaching, expect to make substantial progress.

In addition, the SFL can handle relatively complex situations with the precision of the S-R mediation language without becoming cumbersome. Application of the SFL to persistent problems has not only provided clarification, by abbreviating arguments, but has helped suggest new questions and reformulate others.

In this paper, the SFL is described, its relationship to the S-R Mediation Language is explicated, a methodology is presented for assessing **what is learned**, and a variety of research questions are (re)formulated in terms of the SFL and the related assessment methodology.

SET-FUNCTION LANGUAGE

The essential difference between the S-R mediation language and the SFL is that the association is the basic unit in the former whereas the SFL makes central the notion of a principle. Principles are symbolically represented as functions, f, which mathematically are sets of ordered S, R pairs of a particular type.[1] Responses are normally symbolized, f(S), so as to indicate their functional dependence on the stimuli.

In Table 1, S-R mediation and set-function formulations of basic learning types are contrasted. The mediation formulations are all in terms of S-R

Insert Table 1 about here.

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associations; the SFL formulations are all based on the notion of a function (i.e. principle). In each case, principles act as rules of the form "If A, then B" where A and B may or may not be concepts (c.f. Gagné, 1964). According to the SFL, a concept is a "degenerate" principle, one in which there is a common response (i.e. the function takes on only one value throughout its stimulus domain). A discrete S-R pair is simply a one-element principle.[2]

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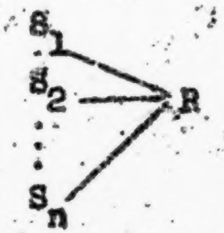
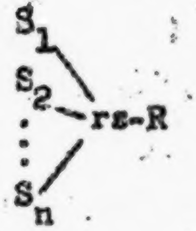
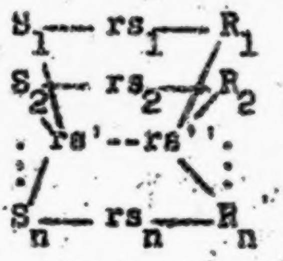
[1] Only those sets of ordered pairs, which satisfy the mathematical definition of a function, are considered here. A set of ordered stimulus-response pairs is a function if and only if each stimulus member is paired with exactly one response.

[2] Examples of each of these learning types are easy to come by (c.f. Gagné, 1964). The classification scheme proposed here, however, is not based on the assumption that the more complex forms of learning (e.g. principles and concepts) are based on simpler types (e.g. associations) as was Gagné's (1964) system. More is said about the dependence of one principle on another in a later section.

Table 1

Mediation and Set-Function Representations of Basic Learning Types*

Single Stage

Type and Description	Mediation	Set-Function
Discrete S-R pair.- A single stimulus-response connection	S-R	$f=\{[S,f(S)]\}$
Concept.-(a) Discrete stimuli connected to same response,		$f=\{[S_i,f_i(S_i)] \mid f_i(S_i)=R; i=1,\ldots,n\}$
(b) A number of multiple cue stimuli connected to the same response by a common rule		$f=\{[S_i,f(S_i)] \mid f(S_i)=R; i=1,\ldots,n\}$
**Principle.- A number of stimuli and responses connected by a common rule	 S_1 rs_1 R_1 S_2 rs_2 R_2 rs'--rs'' S_n rs_n R_n	$f=\{[S_i,f(S_i)] \mid i=1,\ldots,n\}$

Multi-Stage

Type and Description	Mediation	Set-Function
Simple Chain.- A chain of discrete stimulus-response connections	S-rs-R	$g\cdot f=\left\{[S,g\cdot f(S)]\right\}$ where $f=\left\{[S,f(S)]\right\}$ and $g=\left\{[f(S),g\cdot f(S)]\right\}$ are independently determinable
***Chain of Principles.- A chain of rules		$\bigcup_{j=1}^{m} g\cdot f_j=\left\{[S_i,g\cdot f_j(S_i)] \mid i=1,\ldots n \text{ and } j=1,\ldots,m\right\}$ where $f_j=\left\{[S_i,f_j(S_i)] \mid i=1,\ldots,m\right\}$ for $j=1,\ldots m$ and $g=\left\{[f_j,g(f_j)] \mid j=1,\ldots,m\right\}$ are independently observable rules

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* The mediation schemata have been simplified by letting the symbol, rs, represent both the mediating response and the response produced stimulus.

** The mediation formulation shown is one possible way of modifying the S-R language so as to represent the notion of a principle. Gagné (1964) has offered a non S-R schema.

*** There are various ways of combining rules, only a set-function formulation of one-type is given. Any mediation representation would be extremely complex.

Certain additional comments on the single-stage paradigms in Table 1 are in order. First, consider the definition of a concept. In some situations, all that is required is for each of several stimuli to be paired with a common response. For other experimental purposes the stimuli must be multi-dimensioned. Rather than representing such differences by using one and two stage S-R association paradigms, the distinction is made in the SFL by considering sets of ordered pairs in which the number of different functions varies. When the stimuli are discrete, different rules are involved, but the response is identical. The rules might be, "If (the stimulus is) *, then (the response is) dog; If blug, then dog; etc." When the stimuli are multi-dimensioned a single rule may suffice -- e.g. "If large, then dog."

Another point of discussion involves the complexity of an S-R association representation of a principle. At least two sets of mediating stimuli and responses (rs) need to be postulated in order to capture the essence of a principle of the form, "If (the stimulus is) large, then (the response is) the name of the stimulus color." The mediators rs' and rs'' in Table 1 represent the relationship between the S-R pairs comprising such a principle. It is certainly reasonable to think of the stimuli (large colored objects) as eliciting a common response rs' (large), of rs' eliciting rs'' (color), and of rs'' eliciting each of the responses (color names). This breakdown does not, however, indicate why the responses belong to their respective stimuli. The second set of mediators; the rs_i; $i=1,\ldots,n$; serve this purpose.[1]

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[1] It is important to note that the rs_i alone, although they tie the corresponding stimuli and responses together, fail to represent the relationship between the pairs. Such relationships are critical to the SFL approach because they provide the basis for the assessment methodology described in the next section.

Recognizing the difficulties involved, Gagné's (1964) original representation of principle learning did not use the S-R language, but rules of the form, "If A, then B," where A and B are concepts. He represented concepts and simple forms of learning in S-R terms. Because of the continuity break between the concept (S-R) and principle (rule) schemata in Gagné's classification system (1964), Tracy Kendler (1964, 322-323) raised the question of whether new properties emerge at the principle level. The SFL formulation shown in Table 1 suggests not only that new properties emerge, but that these properties can form a basis for simpler forms of learning as well.

The crucial argument in favor of using the rule, rather than the association, as the basic behavioral unit is that of simplicity. S-R representations of principle learning (let alone the learning of several principles related in various ways) are cumbersome and are not likely to be useful in dealing with research questions on complex learning and teaching.

One way of combining principles (i.e. discrete pairs, concepts, or principles) is by chaining. A simple chain like "army-navy-sailor" can be represented, $[S, g\cdot f(S)]$, where $[S, f(S)]$ and $[f(S), g\cdot f(S)]$ correspond to "army-navy" and "navy-sailor," respectively.

Formulating a verbal chain in the SFL suggests a new direction for research in that area. The principles, f and g, could each consist of only one pair. On the other hand, f might be of the form, "If the word-stimulus refers to one end of a scale, then give the word which refers to the other end" or "If the stimulus refers to land, then give an analogous word which refers to water." Thus, f might consist of the pairs, (army, navy), (hot, cold), (tall, short),. . . or the pairs , (army, navy), (airport, aricraft carrier), (continent, ocean), ... The principle, g, also could have any one of several referents. If the operating principle could be identified prior to learning a test list, rather specific predictions could be made. The assessment methodology described in a later section would have relevance in this regard. Another possibility is that the principle referents themselves are manipulable (e.g. by experimental instructions).

What appear to be still more complex situations often yield to SFL analyses. Consider a task like stating a rule for finding the sum of particular arithmetic series. The initiating stimuli are the series, the mediating responses are particular series types (e.g. those series consisting of consecutive odd numbers beginning with one) and the responses are the formulas that can be used to find the sum of any series of that form. The task may be represented by $\bigcup_{i=1} g_i f$ where f is the rule, "If the series is arithmetic (i.e. there is a common difference between terms), then the type of series depends on the first term, the last term , and the magnitude of the common difference," and g_1, g_2, ... are, respectively, "If 'odd,' then N^2," "If 'even,' then $N^2 + N$," ...[1]

In effect, classical learning types can be represented in the SFL by specific kinds of principle and chains of principles.

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[1] There is a one-stage formulation of this situation in which the stimuli again are arithmetic series and the common response is the general formula, $\frac{(A + L)N}{2}$, for finding the sum of any arithmetic series.

PRINCIPLE RELATIONSHIPS

A scheme is presented in Figure 1 for classifying relationships between principles involving stimuli with the same dimensions. To illustrate each of these relationships consider a set of stimuli in which the attributes are size, color, and shape. Let the two principles be "If large, then color" and "If small, then shape." Since it is impossible for an object to be both large and small these principles have no instances in common and are said to be discrete. The principles, "If triangle, then color" and "If large, then color" have some instances in common, those with large triangular stimuli. Each principle also has additional instances of its own -- e.g. small triangles and large circles, respectively. Such principles seem adequately described as overlapping. The principle, "If triangle, then color" is more general than the principle, "If triangle and large, then color" since the former includes all instances of the latter plus some of its own. Such principles are ordered.

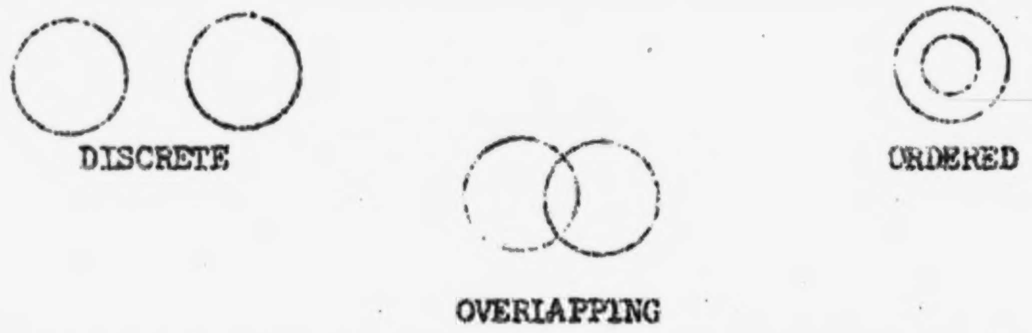

Fig. 1. Three types of relationships between principles. Each circular region represents an identifiable principle.

Generality.- Scandura, Woodward, and Lee (1965) have shown that the behavior induced by presenting statements of ordered principles conforms to expectations. Two experiments were conducted, the independent variable in both cases being principle generality.

In the first experiment, each group of 17 college Ss was presented with one of three ordered principles dealing with a number game.[1] The application of each principle was illustrated with a common game. The least general principle (S), adequate for winning only $(\bar{7},31)$ games, was stated, "···make 3 your first selection. Then ··· make selections so that the sums corresponding to your selections differ by 7." Principle (SG) was adequate for solving $(\bar{7},i)$ games i=1, 2, ···, n and was stated, "the first selection is determined by dividing the desired sum by 7. ··· Then ··· make selections so that the sums corresponding to your selections differ by 7." The most general principle (G) was adequate for solving (j, i) games j=1,2,···,m; i=1,2,···,n and was stated, "the first selection is determined by dividing the desired sum by the one more than the largest integer [in the set from which the selections must come] ··· Then ··· make selections so that the sums corresponding to your selections differ by one greater than the largest integer [in the set from which the selections must come]."

All Ss, including two control groups, were tested on three problems. The first was within the scope of each principle, the second within the scope of all but the S rule, and the third only within the scope of principle G. The results were straightforward. Of those 15 Ss in group S who solved problem one, none solved problem two, and only one solved problem three. The corresponding numbers for groups SG and G were, respectively, 5, 4, 0 and 5, 4, 4. Within the scope of each principle there were only chance differences in performance on the problems. On the other hand, only one S solved an extra-scope problem.

The second experiment was conducted with junior high school Ss and dealt with arithmetic series. In addition, the presence or absence of an illustrative example was varied independently of principle generality. The pattern of results in this experiment was almost identical to that in the preceding experiment. None of the *Ss* solved a problem beyond the scope of the principle presented, whether an example was or was not given. Fewer (3) of the 19 G-with-example *Ss*, however, solved the most general problem (three) whereas 18 solved problem one and 14 solved problem two ($p < .004$). This was attributed to a conceptual difference in the means required for determining the number of terms in the third problem series and in the others, including the example.

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[1] In the game, two players alternately select numbers from a specified set of consecutive integers (including 1) and keep a running sum. The winner is the one who picks the predetermined last number in the series. If this sum is 31 and the set consists of the integers 1-6 the players select numbers from 1-6 until the cumulative sum is either 31 or above (in which case no one wins). There is a compound rule which allows the player, who goes first, to win any such game. "Divide one more than the largest number in the set into the desired sum - make the remainder the first choice - on subsequent tries, consistently select that number which when added to the opponent's preceding choice sums to one more than the largest number in the set." There are actually two principles involved, one involving selection of the first number and the other involving the sequence choices. To win consistently, both principles may be employed, one after the other. The acquisition of each principle can be determined by analyzing distinct aspects of the overall game plan.

This game is called Nim (see Parks, 1964, 55-58).

Nonetheless, these results certainly justify the ordering of verbally presented principles as to generality, particularly when possibly interfering examples are *not* used.[1]

Abstractness.- In addition to transfer potential, some of the principles differed as to *learnability*. The S rule, in experiment one was significantly easier to learn, as judged by problem one performance, than were the rules SG and G ($p < .007$ in both cases). There was essentially no difference in learning rules SG and G. We anticipated finding a difference, due to generality, between the latter groups as well and, originally, we were tempted to attribute the lack of such an effect to scale insensitivity in the lower ranges.

Further analysis of the situation, however, indicated that the rules may have differed as to interpretability. Two possible explanations suggested themselves. The rules SG and G were both more complex, in the sense that there was more to remember, and more abstract, in that comprehending (i.e. applying) a (sub) rule for determining a number has more prerequisites than comprehending a specific number.

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[1] It should be noted that the rules were not stated in a form which would allow the *Ss* to discriminate between problems where the rule was and was not appropriate. In effect, all of the rules were of the form, "There is a pattern to the game which will enable you to win whenever you are allowed to make the first selection. You must, however, make an appropriate first selection and then proceed according to a specific pattern." This procedure made it possible to also obtain information on response consistency which is discussed in a later section (Assessment Methodology).

In experiment two, 8, 5, and 3 of 20 Ss in the S, SG, and G groups-without-example, respectively, were successful on problem-series one.[1] But, only the S-G difference approached significance ($.05<p<.10$). Frankly, we were surprised that only 8 of the Ss, presented with the rule, 50 x 50, gave the correct sum (2500) of the first test-series. The fact that all Ss except those in the experiment two S group consistently applied the rule taught to each problem (whether appropriate or not), however, gave us a clue. Whereas all 5 (7) of the SG (G) Ss, in experiment two, who used the rule taught on problem one also used the rule on problems two and three, only one of eight Ss in the S group did so. These Ss apparently recognized the inapplicability of the rule and failed to respond.[2] Such reluctance may also have been evident on problem one. Questioning the Ss and scrutinizing the test papers suggested that this was a feasible interpretation.

What we failed to do and what needs to be done is to deal explicitly with statement complexity and abstraction. Specifically, it would be extremely desirable to have some means of preclassifying these characteristics.[3]

In any case, I feel that prior learning (e.g. Gagné, 1962; Scandura, 1965_a, 1965_b) is likely to be a crucial factor. It may, in fact, be impossible to give a behaviorally meaningful definition of either complexity or abstractness without explicit reference to prerequisite knowledge.

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[1] Almost all of these Ss shown the example were successful since test-series one was identical with it. This was not the case in experiment one in which all of the Ss were exposed to an instance of the (7, 31) game. There are many variants of the (7, 31) game.

[2] It can probably be assumed that most junior high school Ss would find it inconceivable that the series 1 + 3 + ··· + 99 and 1 + 3 + ··· + 49 have the same sum (50 x 50).

[3] Indeed, with regard to complexity I would not be surprised to see Miller's (1956) magic number seven again raise its familiar head.

Since I have gone this far, let me go a little further and suggest a behaviorally relevant definition of abstractness. One statement of a principle is more abstract than another if the set of prerequisite principles of the former properly contains the corresponding set of the latter. That is, the more abstract the principle statement, the more prior knowledge is required. According to this definition, "If (given an) odd series, then the sum of the series is obtained by computing N^2 where N is the number of terms in the series," is more abstract than "If odd series, then the sum of the series is obtained by multiplying the number of terms in the series by itself." Being able to give the appropriate product when told to compute N^2, where N = specific integer, is prerequisite to applying the rule when stated in the former manner. The latter statement makes no such requirement.

According to this definition, the representations of a particular principle can only be partially ordered according to abstractness. Figure 2 indicates how various representations of the principle, N^2, may be placed on an abstraction lattice. The middle representation on the left and the one on the right are not directly comparable. In the former case, S needs to understand the symbolism N^2 (i.e. N X N) and, in the latter case, S needs to know how to find the number of terms in an odd arithmetic series. Some of our recent experiments (Scandura, 1965a, 1965b) suggest that unless S can operationally make use of a term, statement, or rule; an explanation of a more complicated rule using these notions is essentially not understandable (i.e. can not be used effectively).

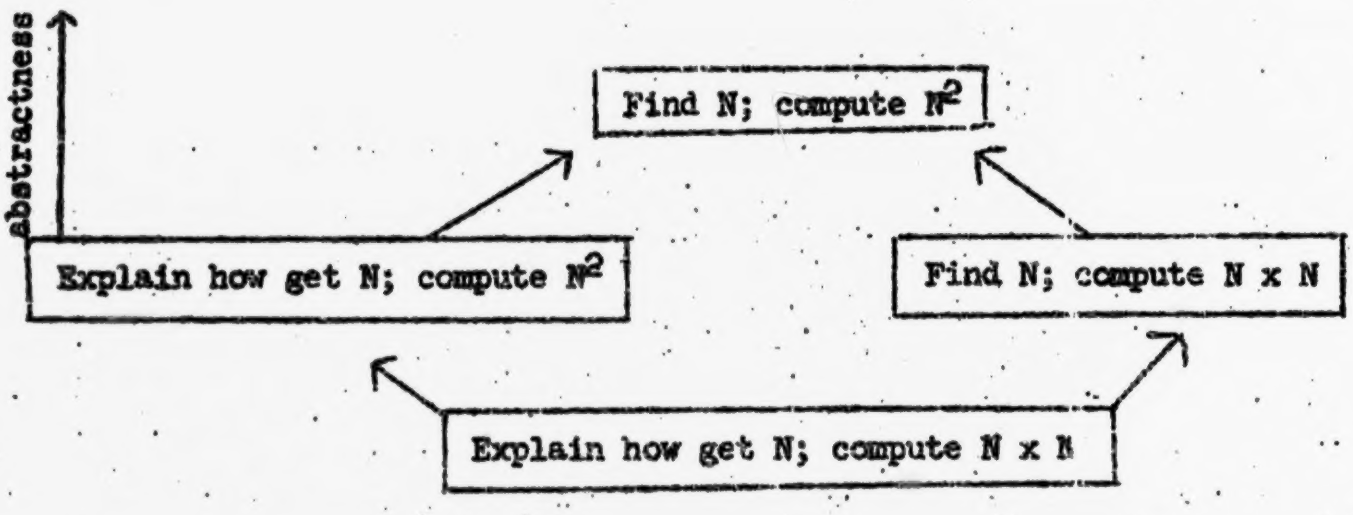
abstractness
Find N; compute N^2
Explain how get N; compute N^2
Find N; compute N x N
Explain how get N; compute N x N

Fig. 2. A rule for obtaining the sum of odd arithmetic series presented at various levels of abstraction. The statements "Explain how get N" and "compute N x N" correspond to descriptive statements, in common place terms, for "finding N" and "computing N^2," respectively.

Conjunction-Disjunction.- So far, reference has been made only to *conjunctive* principles, principles of the form, "If A *and* B, then C." As with concepts (e.g. Bruner, Goodnow, and Austin, 1956), *disjunctive* principles are of the form, "If A *or* B, then C." The set properties of principles, however, indicate that every disjunctive principle can be expressed as the union of two or more conjunctive principles with a common "then" clause. Thus, the union of the principles, "If A, then C," and "If B, then C," includes exactly the same instances as does the disjunctive principle, "If A or B, then C."

In an entirely analogous fashion, disjunctive concepts (e.g., Bruner

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et al, 1956) correspond to two or more conjunctive concepts with a common response. When looked at in this fashion it is not hard to see why disjunctive concepts are harder to learn than conjunctive concepts. There are more concepts to learn and it may be harder to learn two or more concepts than it is one.[1]

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[1] Symbolic logic can be used to symbolize the relationships involved in conjunctive and disjunctive principles. For example, principles of the form, "If A and B, then C and D," "If A or B, then C," and "If A, then B or C," may be symbolized as:

(1) $(A \cdot B) \Rightarrow (C \cdot D)$,

(2) $(A \vee B) \Rightarrow C$, and

(3) $A \Rightarrow (B \vee C)$, respectively.

This use of the statement logic is similar to that by Anderson (1964) to analyze S-R mediation theory. The letters A, B, C, and D, may refer to stimulus cues and dimensions or to nominal stimuli and responses. We deny the need to consider principles of the form, 3 (see footnote p. 21). It is beyond the scope of this paper to more fully consider the relationships between the SFL and the statement logic.

NOMINAL AND FUNCTIONAL STIMULI

So far, stimuli have been treated as unitary elements. Their multidimensional nature has been considered only incidentally. In stating the principle, "If large, then color," for example, reference is made to both the size and color cues of the eliciting stimulus.

In the S-R language, a distinction is often made between nominal and functional stimuli, the nominal stimulus referring to that physically present and the functional stimulus to that aspect of the stimulus which determines the overt or implicit response.

In the SFL, in which the principle becomes the fundamental unit, not only is the nominal-functional distinction maintained, but functional stimuli are classified as to role. A functional stimulus may serve to cue a principle (e.g. large) or a response (e.g. color). The former is denoted a "principle identifying cue" and the latter a "response determining attribute."[1]

A principle identifying cue(s) is common to the stimulus members of each instance of the principle. A response determining attribute or combination of attributes varies 1-1 with the response members when all other determining attributes remain constant. Consider a principle represented by a statement of the form, "If a, then B and C" where *a* refers to the common physical stimulus property and *B* and *C* refer to dimensions (i.e. classes of physical stimulus characteristics). Each of the principle related stimuli have the property a and one value of the attributes B and C. The responses, whatever they are, vary with B and C. If the value of B (C) is held constant, then the responses depend uniquely on C (B). With respect to the principle, "If the stimulus is large, then the response is determined by the color and shape," a corresponds to large, B to color, and C to shape.

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[1] The S-R language considers only response determining cues and, in fact, there is no apparent need in the S-R language to make the above distinction (except possibly in discrimination learning). Principle and response identifying cues are *identical* in simple association and concept learning types. With respect to the pair, [*, Rug.], * probably serves both functions. In concept learning, involving multi-dimensional stimuli, the functional stimulus (e.g. "red") also serves to both cue the principle and the response.

The responses may be merely the names of the color and shape properties or they may depend on these attributes in a more subtle fashion. For example, the principle might have been,

(1) "If a series is arithmetic,[1] then the response is determined uniquely by the first term in the series (A), the last (L), and the number of terms (N) according to the formula $\left(\frac{A+L}{2}\right)N$."

The stimulus attributes (A,L,N) again determine the responses, but they exert this control via an algebraic rule rather than by "naming" colors and shapes.

Notice that verbally stating principle (1) and having S learn to repeat the definition verbally would not guarantee that S could use the principle. Being able to identify instances of a principle and to determine and combine the response identifying values are prerequisite to using (i.e. "understanding") the principle. This undoubtedly is what people like Bruner (e.g., 1961, 29) have in mind when they speak of the learner acquiring a sort of verbal glibness without true understanding.

It is also worth noting that the generality of a principle depends on the number of stimulus dimensions that vary with the responses (i.e. the number of response identifying attributes) and the number of principle identifying cues. Thus, the principles, "If large and *black*, then (the response varies with) shape" and "If large and *white*, then shape," may be considered special cases of the principle, "If large, then (the response varies with) *color* and shape." The latter, more general, principle at once has fewer identifying cues and more response determining dimensions than the other two, more specific, principles. In effect, it appears that critical response dimensions are traded off with critical principle identifying cues. The more general the principle, the more stimulus attributes vary with the responses; the more specific the principle, the more stimulus properties are required to identify the principle. The total number of critical properties remains constant.[1]

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[1] In this case, a common difference between terms would be the principle identifying cue.

An example from the "real world" indicates the non arbitrary nature of this invarient. Consider the principle, "If an arithmetic series has a common difference of 2, then the sum is given by $(A + N - 1)N$. The principle identifying cue (within a population of arithmetic series) is that of a common difference of *two* between adjacent terms. The responses vary with A and N (where A is the first term of the series and N is the number of terms).

The stimuli within the scope of this principle are all arithmetic series having a common difference of two. They differ as to the *first* term of the series and the *number* of terms. A special case of this principle might involve, for example, only those series beginning with the number one (an additional identifying cue). Here, the sums (i.e. responses) would vary exclusively with the number of terms, as given by the formula $(1 + N - 1)\ N = N^2$. Other special cases can be similarly derived. The essential point, again, is the constant number of critical stimulus properties.[1]

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[1] I have said nothing about the need for principles of the form, "If A, then B or C." I am of the opinion that such probabalistic response determination is more apparent than real and results, primarily, from our inability to identify those stimulus properties, D and D', distinguishing the principles, "If A and *D*, then B," and, "If A and *D*', then C."

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[1] This invarience can be stated and easily proved as a theorem. First, some definitions are required:

(1) A stimulus cue is said to help identify a principle if the cue is common to all stimuli within the principle domain,

(2) A stimulus dimension or combination of dimensions (when dimensions are correlated) is said to help identify a response if the values of this dimension vary 1-1 with the responses when all other stimulus dimensions remain fixed,

(3) P is a partition of a principle, f, if $P = \{f_i / i=1,\ldots n\}$ where

$$\bigcup_{i=1}^{n} f_i = f \text{ and } f_i \cap f_j = \begin{Bmatrix} \emptyset & \text{if } i \neq j \\ f_i & \text{if } i = j \end{Bmatrix}.$$

In words, a set of principles, P, partitions another principle, f, if an S-R pair is an instance of f if and only if it is an instance of one of the principles $(f_i, i=1,\ldots,n)$ in P and no S-R pair is an instance of more than one of these principles,

(4) A partition $\bar{P}$ is said to be nonredundant if for each f_i, $i=1,\ldots n$, a constant number of the response identifying dimensions of f remain constant over f_i. If only irrelevant dimensions of f are held constant, the partition is said to be redundant.

The theorem and its proof follow directly from these definitions.

Theorem. Given a principle defined over a set of stimuli, then any nonredundant partition of f preserves the number critical stimulus properties.

Proof: By definition of a nonredundant partition, at least one of the response identifying dimensions remains constant. Suppose k dimensions remain constant. Then, by definition, there are k more principle identifying cues for each principle in the partition. Therefore, the number of critical cues is invarient.

Actually, a complete formulation of this problem requires that a precise definition be given to the notion of a dimension (c.f. Restle, 1961). This is beyond the scope of the present paper.

KNOWLEDGE ASSESSMENT

What is learned. Commitment to a stage-approach to research on complex learning and teaching necessitates attention to knowledge assessment. Suppose, for example, that stage-one consists of learning the list shown in Figure 3. These pairs could be learned as four distinct single-pair principles or as instances of two or more general principles, "If triangle, then color" and "If circle, then size." Such differences in stage-one learning could critically affect learning or performance during stage-two.

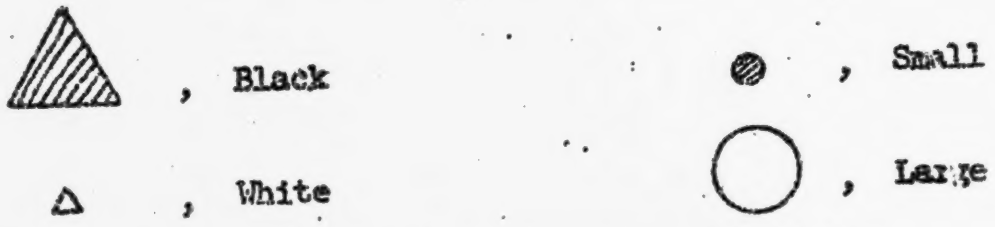

Fig. 3. Four S,R pairs involving two principles.

In this section, the concern is with a methodology for assessing *What is learned*. The general approach is an outgrowth of some of our earlier research. Greeno and Scandura (1965) found that in a verbal concept learning situation, *S* either gives the correct response the first time he sees a transfer stimulus or the transfer item is learned as its control.[1]

The present author later reasoned that if transfer obtains on the first trial, if at all, then responses to additional transfer items, under certain conditions should be contingent on the response given to the first transfer stimulus. In effect, a first transfer stimulus could serve as a test to determine what had been learned during stage-one, thereby making it possible to predict what response S would give to a second transfer stimulus. The results of a pilot study were revealing. In those cases where transfer potential was indicated, the responses to the second set of stimuli conformed to prediction in 48 of 52 cases. In this study, S was allowed to make a response which indicated that no relationship was noted between the learning and test materials. Without this control, correct responses to a transfer stimulus would have occurred by chance in about 1 out of 4 cases.

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[1] The stimuli were Underwood and Richardson (1956) nouns, the responses were nonsense syllables, and the lists were learned by a self-paced anticipation method.

A similar methodology was used in the case of principle learning -- and the results were identical. Whenever the Ss responded to one test stimulus in accordance with a principle, they also responded, according to the principle, on subsequent stimuli.

For illustrative purposes, again consider the pairs shown in Figure 3. Testing, after the pairs are learned, for the acquisition of the principle, "If triangle, then color," might consist of presenting the stimulus, △. The response "white" would indicate that the principle was operating. The anticipated response to, ▵, would be "black." If the stimuli were outside the principle domain or, indeed, if the principle had not been acquired, no such prediction could be made.

There are two major questions that can be asked of the assessment methodology described above. Under what conditions can type of learning be validly assessed and under what conditions does response consistency obtain? The utility of the assessment procedure is largely dependent on the ability to control such factors.

In order to answer the first question, the effective variables and an acceptable criterion need to be identified. One such set of variables concerns the instructions given before the original learning and between the original learning and the first test. A feasible criterion, against which to compare learning type, defined in terms of a test response, is S's verbal report as to how he learned the items and/or as to the basis for his test responses.

The predictive value of the methodology described is dependent on knowing when S will employ the same responding set. Consistency of response to the test-stimuli may be influenced by feedback as well as instruction variables operating between the first and second test responses. The effects of positive, negative, and neutral reinforcement of the first test response may be crucial. Telling S how he should respond or indicating that the "rules have changed," by hint or choice of test stimuli, may also affect response consistency.[1] On the other hand, suggesting that the first response is appropriate may encourage use of the same responding set on the second test. In our pilot experiments, S was told that he was correct regardless of how he responded to the test stimuli. He also was encouraged to respond on the basis of his prior learning. In effect, the experimental situation was designed to both control and capitalize on Einstellung for assessment and predictive purposes. Under these conditions, response consistency, based on both principle and concept learning, was near perfect.

Contrast the variables described here with those more typically manipulated in psychological settings. The latter variables, to which we refer, are those which might affect the *probability* of principle learning. In the situations described, such variables become boundary conditions to be altered so that the desired proportion of *Ss* attain a particular principle. The assessment methodology described was designed solely to determine "what is learned" and when that learning affects performance.

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[1] In the study by Scandura, Woodward, and Lee (1965), many of those junior high school Ss who were presented with the rule "50x50" apparently recognized the inapplicability of the rule to the transfer series. They failed to persist in using the rule taught whereas the other groups were highly persistent. The reason for this difference was attributed, by the authors, to prior learning which indicated to the Ss that the rules, indeed, had changed.

The assessment problem, of course, is not always as simple as has been depicted. It has already been pointed out that different principles frequently have instances in common (e.g. ordered and overlapping principles). In order to determine which one of two (or more) principles are operating, it is essential that the test pair belong only to the principle in question.

With actual subject matters, additional factors are involved. In the first place, it is not always easy to specify uniquely the basis for an overt response. There is often more than one path to the goal. As an illustration, consider a situation[1] in which *S* is asked to compute 35·449 + 35·551 as rapidly as possible. *S* can laboriously multiply 35 times 449 and 35 times 551 and then add the products or he can recognize this as an instance where the distributive principle would allow him to compute 35(449+551)=35·1000=35,000. Clearly, it is not the response alone which determines what is learned (i.e. the "way" in which the problem is solved), but the *time* it takes to respond. If the correct answer is given in a short time, the distributive principle was probably used. Giving the correct answer in a relatively long time would likely indicate the usual computational rule. If *S* gives an incorrect answer or if the problem is so easy that there would be little time differential no matter how *S* does the computations, further complications are introduced.

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[1] Such a situation occurs in the University of Illinois School Mathematics Program (Jack Easely, personal communication).

In order to better specify the source of an observable behavior, the careful selection of test stimuli and responses is essential. Ideally, these elements should not only indicate the likelihood of the particular responding set in question, but should eliminate all other alternatives. Although probably not attainable, this ideal can be approached in many cases.

Another problem involved in work with actual subject matters is that of complexity. More than one principle may, and usually does, enter into a single test response. To determine the learning underlying the response, it is often necessary to individually assess for each principle. In other cases, it may be sufficient to simply test for the acquisition of the conjunction of principles; aspects of the compound response often provide information about the elementary principles. For example, consider a game in which the two players alternately select numbers from a specified set of consecutive integers (including 1) and keep a running sum, the winner being the one who picks the last number in a series with a predetermined sum. If this sum is 31 and the set consists of the integers 1-6, the players select numbers from 1-6 until the cumulative sum is either 31 or above (in which case no one wins). There is a compound rule which allows the player, who goes first, to win any such game, "Divide one more than the largest number in the set into the desired sum - make the remainder the first choice - on subsequent tries, consistently select that number which when added to the opponent's preceding choice sums to one more than the largest number in the set." There are actually two principles involved, one involving selection of the first number and the other involving sequence choices. To win consistently, both principles may be employed, one after the other. The acquisition of each principle can be determined by analyzing distinct aspects of the overall game plan.[1]

In many test situations, there are few available responses from which to choose (as in True-False and Multiple Choice tests). Under these conditions, there are additional problems of assessment since there is a high probability of giving any particular response (by guessing) irrespective of learning type. A similar problem obtains in assessing concept learning. There are at least two ways of overcoming this problem: (1) present more than one test stimulus and (2) include appropriate controls (e.g., Greeno and Scandura,(1965).

How to learn.- In addition to assessing what is learned, the problem of assessing learning process may also be considered. Although the problem is considerably more complex, the methodology involved is but a simple extension of that used to assess learning type. In addition to presenting to-be-learned material and determining by assessment, what is learned, a second set of materials must be presented. Assessing what is learned, in the second case, would then provide a basis for comparing the manner in which the original and test (second) displays are encoded. This paradigm can be represented by:

Learn A, present A test stimulus, learn B, present B test stimulus whereas the original assessment paradigm is represented:

Learn A, present A test stimulus one, (present A test stimulus 2).

More is said about this problem in the next section on applications.

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[1] This problem has much in common with diagnostic work. There are also strong similarities with Gagné's (1962) approach to task analysis. Although it is beyond the scope of the present paper to show that it is so, there is strong reason to believe that the SFL can be used to make these procedures more explicit. Such precision may be a necessary adjunct to more sophisticated diagnostic technologies.

APPLICATIONS

In this section, a variety of persisting problems are formulated in SFL terms. In most cases, the assessment methodology also plays an important role. Problems involving paired associate (PA) principle learning, expository and discovery modes of instuction and cognitive development are considered.[1]

Principle Learning.- The question of relationships between S-R pairs seems so obvious, and so basic, that one wonders why it has not been studied extensively. Because it provides a simple context in which to contrast mediation and set-function formulations, the problem is described in some detail.

Consider a PA context in which the relationships between four pairs are varied while the other factors are held constant. In Figure 4, such a manipulation is accomplished by selecting two principles, "If the stimulus is black, then the response is the name of its shape" and "If white, then size." The exper-

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Insert Figure 4 about here.

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[1] Other areas of potential application include human performance (e.g. Posner, 1964) and task analysis (e.g. Gagné, 1962).

Responding to the stimuli in Figure 4 would constitute an information conservation task according to Posner's (1964) classification scheme since there are four stimuli and four responses. According to a previous discussion, such performance could, however, be based on the acquisition of four one-instance principles or two two-instance principles. In the latter case, the task could conceivably act more like one of information reduction (Posner, 1964). In effect, performance (latency) could depend on the learning underlying the performance. It is beyond the scope of this paper to go further into this area, but the potential implications are clear.

The task analysis procedure of breaking down skills into components seems to

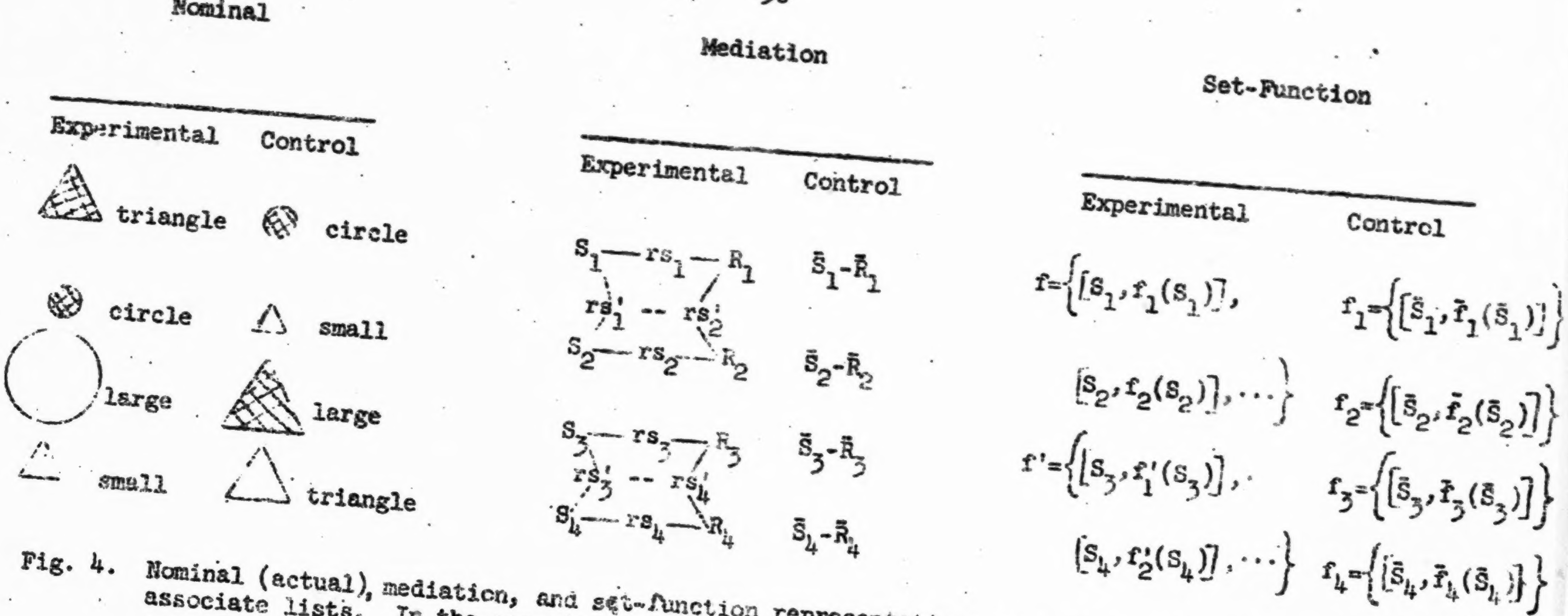
Nominal
Experimental
Control
triangle
circle
circle
small
large
large
small
triangle
Mediation
Experimental
Control
S_1 — rs_1 — R_1
rs'_1 -- rs'_2
S_2 — rs_2 — R_2
S_3 — rs_3 — R_3
rs'_3 -- rs'_4
S_4 — rs_4 — R_4
$\bar{S}_1-\bar{R}_1$
$\bar{S}_2-\bar{R}_2$
$\bar{S}_3-\bar{R}_3$
$\bar{S}_4-\bar{R}_4$
Set-Function
Experimental
Control
$f=\{[S_1, f_1(S_1)], [S_2, f_2(S_2)], \ldots\}$
$f'=\{[S_3, f'_1(S_3)], [S_4, f'_2(S_4)], \ldots\}$
$f_1=\{[\bar{S}_1, \bar{f}_1(\bar{S}_1)]\}$
$f_2=\{[\bar{S}_2, \bar{f}_2(\bar{S}_2)]\}$
$f_3=\{[\bar{S}_3, \bar{f}_3(\bar{S}_3)]\}$
$f_4=\{[\bar{S}_4, \bar{f}_4(\bar{S}_4)]\}$

Fig. 4. Nominal (actual), mediation, and set-function representations of experimental and control paired-associate lists. In the experimental list the pairs are interrelated; in the control list, they are not.

Broken lines indicate associations to be learned; solid lines indicate previously learned associations. Two principles are involved (in the experimental list): (1) If black, then shape, (2) if white, then size. The symbols, rs, refer to both the mediating response and the response produced stimulus. rs'_1 corresponds to "black", rs'_2 to "shape", rs'_3 to "white", rs'_4 to "size", rs_1 to triangle property, rs_2 to circle, rs_3 to large, and rs_4 to small. Without this notation simplification, the mediation interpretation would be more complex.

According to the SFL, the experimental list can be learned as either two (If black, then shape and If white, then size) or four discrete principles whereas the control list can only be learned as four. Thus, f_1 and f'_1 may or may not be the same as f_2 and f'_2, respectively. $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$, and $\bar{f}_4$ are necessarily distinct.

imental stimuli, corresponding to these principles, are all either black or white and have some well-known (by the Ss) shape and size. The responses are simply the shape and size names. In the experimental list, two pairs correspond to each of the two principles. Note that the control responses and stimulus properties (i.e. size, color, shape) are identical with those in the experimental list. In addition, the responses, in both cases, are names of one of the properties of the corresponding stimuli. Any differences in the learnability of these lists would be hard to attribute to anything but the presence of relationships between pairs in the experimental list.[1]

The mediation description of the list contingencies, even as modified, leaves much to be desired. The representation of principle learning is relatively complex and would have been even more so had we not let the rs represent the typically made distinction between the mediating responses and their assumed stimulus properties. A more crucial limitation is that the chain diagram simply would not make clear why R_1 is the overt response to S_1, rather than R_2, without the addition of the more direct two-stage chains involving the rs_i $(i=1,\ldots,4)$. As indicated previously, this 1-1 pairing does not follow from an analysis of the

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be quite analogous to determining those learnings prerequisite to a particular principle. The reader is referred to Gagné's writings (e.g. 1962,1964,1965) for a more detailed account of these ideas.

[1] It may appear that an appropriate control list, could be constructed by pairing the same stimuli and responses in random fashion. Alas, this turns out not to be a critical control. Any differences between the groups can be attributed to pre-experimental associations between stimulus properties (e.g. shape) and the corresponding responses (e.g. shape "names") rather than to relationships between pairs. These considerations dictate that prior learning be either absent or equivalent in both groups. The SFL requires that none of the control pairs have any (obvious) relationships (i.e. no principle includes more than one pair). The experimental lists shown in Fig. 4 meet these requirements.

S-R links in the chain.

In view of the increasing difficulties implied in dealing with still more complex problems, we cannot help but recall pre-Copernican epicycles and related attempts to salvage geocentric theory. How much simpler when the facts are expressed in the SFL.

Judith Anderson and I conducted a pilot study that is relevant. Its purpose was to determine: (1) the effects of the number of instances per principle on the probability of principle learning and on the rate of learning and (2) relationships between principle learning and learning rate.

The materials to be learned consisted of lists of 12 pairs similar, in type, to those shown in Figure 5. Each stimulus had a property relating to shape, borders, shading, outline, and color. Four colors and eight values of the other four attributes were used to make a total of 32 stimuli. The responses were labels attached to one of the non-color stimulus properties. Of the 12 pairs in each list, six were instances of one principle (P6), three were instances of another (P3), two were instances of a third (P2), and one was an instance of a fourth (P1). In each case, the principle identifying cue was a color and the response determining class was either shape, borders,

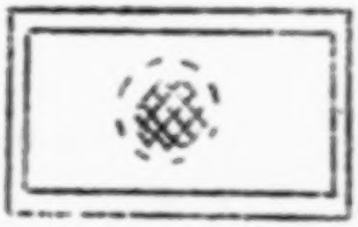
STIMULUS	RESPONSE
	dashed

Fig. 5. This sample pair is circular, shaded with crosshatches, outlined in dashes, has two borders, and is black.

shading or outline. The four identifying cues and determining classes were randomly paired to form four principles (e.g., If black, then shape), which appeared equally often under each condition.

The PA list was learned by the anticipation method to a criterion of three consecutive errorless trials. To determine whether the principles had been acquired while learning the pairs, each S was shown two additional lists of four stimuli each. In both lists, one stimulus was associated with each principle and each stimulus appeared only once. Responding according to the principle was presumed to indicate that the principle had been learned.

Prior to learning the original list, each of the 20 college Ss was pretrained so that he was familiar with the stimulus dimensions and could name each stimulus property. These responses were typed on a card and were always available to S. In addition, S was told that a pattern was involved which might facilitate his learning and guide his responses to the transfer stimuli.

The dependent variables were the average number of errors per instance for each S (on each of the four principles) and the number of appropriate responses to the test stimuli (two for each principle).

Except for a very small reversal between treatments P3 and P2, the average number of errors per instance decreased with the number of instances per principle: 5.0, 3.4, 3.5, and 2.7, respectively ($F=4.029$, $df=3/76$, $p<.05$). The difference between P1 and P2 was significant ($F=5.358$, $df=1/76$, $p<.05$) but none of the other adjacent means differed significantly. Apparently, the rate of learning increased sharply with the addition of a second instance and then tapered off.

The number of appropriate responses to the transfer stimuli was also affected by the number of instances per principle. There were 27, 8, 15, and

9 appropriate responses given to P6, P3, P2, and P1 transfer stimuli. Although the trend was not entirely regular, a sign test indicated that the degree of principle learning was higher in treatment P6 than in the average of treatments P3, P2, and P1 ($Z = 2.6$, $p < .005$).[1]

Another analysis demonstrated that learning rate was related to P6 principle learning. Of those 9 Ss who responded appropriately to both P6 test stimuli,[2] 7 had below median (2.61) error scores, indicating more rapid learning; of those 11 Ss who responded appropriately to at most one test stimulus, 8 had above median error scores. An exact test (Finney, 1948) indicated a significant relationship between principle learning and learning rate ($p < .035$).[3]

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[1] It might be argued that the difference in the number of appropriate responses was due to there being more responses per category (e.g. shape) in treatment P6. When in doubt, the Ss may have tended to give a response from the most frequently experienced category. A comparison of the average number of P6 responses given to the P3, P2, and P1 transfer stimuli (16) was not significantly higher than the ten P3, P2, and P1 responses given to the P6 stimuli ($p > .10$).

[2] It should be noted that the probability of giving two appropriate responses in a row by chance-guessing is one out of 16. This fact precluded the possibility of obtaining significant relationships with respect to the other principles. Only 3, 5, and 2 Ss gave both desired responses to the P3, P2, and P1 test stimuli, respectively.

[3] It is interesting to note that Erickson (1963), in studying the von Restorff effect in a paired associate task, also obtained results which were difficult to explain in terms of relationships between stimuli and responses (i.e. stimulus and response discrimination) rather than between pairs. Erickson (1963) found that an S-R pair which differed from the others in the list only in terms of the relationship between the stimulus and the response was learned faster than the other pairs.

Exposition and Discovery.- Previous studies involving expository and discovery modes of instruction have not been entirely consistent, even when the studies apparently have been well controlled (e.g. Craig, 1956; Gagné and Brown, 1961; Haselrud and Meyers, 1958; Kersh, 1958; Wittrock, 1963). Part of the difficulty has been due to a lack of consistent terminology (e.g. see Wittrock, 1963), but other problems are not so easily disposed of. Thus, Wittrock (1963) found that rule given groups performed better on a transfer test than did a discovery group. In the Gagné and Brown (1961) study, however, the results were reversed.

In explaining the results of such experiments, recourse is frequently made to "what is learned" (e.g. Gagné and Brown, 1961; Scandura, 1964; Wittrock, 1963). Unfortunately, such explanations are not only post hoc but are without objective evidence independent of the experimental results to be explained. A more plausible alternative may be to make explicit an a priori distinction between logical and behavioral factors.

Consider first some of the experimental factors involved in learning by exposition. Frequently, a rule is presented directly and the Ss are tested to see if they can give appropriate responses to stimulus exemplars of the rule. The scope or generality of the rule, however, is rarely made explicit. This has not often caused real difficulty since scope normally is determined rather directly. Still it is important to note that scope is a logical variable dependent entirely on relationships between the rule and its instances. There are no behavioral questions involved.

The logical factors involved in discovery treatments are more subtle; yet, they are at least as important. Teaching by discovery usually involves presenting one or more instances and/or test stimuli and/or cues, hints, and other indirect guidance as to how *S* should "process" information. In one form of discovery, *S* is presented, in turn, with several instances of a principle and then is asked to give the correct response to the stimulus member of a new instance. In effect, the learner is required to abstract a principle, much as he would in concept learning. In another form of discovery, the learner is shown only stimuli and is given direction as to how he might determine the correct response. In both cases, the "discovery" of principles also necessarily involves learning *how to acquire* principles. The generality of the cues and hints given would determine the range of applicability of the processing mode and the number and nature of such cues and hints would determine the likelihood of acquiring such a mode. At one extreme, this could involve learning to acquire a simple principle with no direction and, at the other extreme, involve being told a higher order principle for determining a broad class of subordinate principles. Again, logical factors appear to be as crucial as behavior variables in determining outcomes.

Unfortunately, there has been little attempt in studies on exposition and discovery, to predetermine relationships between test stimuli and responses and presented information. The folly of not doing so becomes increasingly apparent as more and more studies, involving complex materials, demonstrate that structural factors can be (and usually are) more important than traditional behavior variables (Gagné, 1962; Scandura, 1965b; Scandura and Behr, 1965; Scandura, Woodward, and Lee, 1965).[1]

Determining relationships between presented and test materials is a logical problem. What S does with the information (i.e. test performance), however, is a behavioral question and depends on a variety of factors, not the least of which is the ease with which the presented material can be learned (Scandura, Woodward, and Lee, 1965). In the study described earlier (Scandura et al, 1965), predicting perfomance was largely a structural problem contingent primarily on learnability.

Nonetheless, most researchers (e.g. Craig, 1956; Haselrud and Meyers, 1958; Kersh, 1958; Kittle, 1957; Wittrock, 1963) have failed to distinguish between variables affecting what learning can obtain and variables affecting whether learning does obtain. Treatments have differed both in their logical relationship to the test materials and in ease of learning. This has not affected the validity of the results, but it has made them difficult to interpret. It is impossible to present really definitive analyses of most studies in this area since there has been no systematic attempt to pre-experimentally specify structural factors. To be definitive, criterion measures, as well as treatments, need to be chosen carefully for specific purposes.

In the Wittrock study (1963), for example, rule (given-not given) and answer (given-not given) were independently manipulated. The rule and answer given group (RA) effectively was presented with two rules -- one specific to one problem and the other of a more general nature. Which rule (if not both) was learned by a majority of the *S*s would depend on their relative ease of learning.[1] The rule (R) and answer (A) given groups had no such choice. Group R was shown the general rule and a stimulus on which to apply it. The A *S*s were shown an instance (S-R pair) of the rule and were given directions to discover it.

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[1] Logical analysis has probably not been so crucial in traditional laboratory studies since, in most cases, relationships between presented and test materials are quite direct.

Rule learning may have been harder than answer learning, but only rule learning made it possible to perform successfully on the learning test which consisted of a new instance within the scope of the rule.

The no rule-no answer (discovery) group was required to discover the rules independently as were the A *S*s. The A *S*s, however, had the advantage of seeing an instance of the rule. The performance of these groups on the learning test gave some indication of the difficulty of the task with which they were confronted. That the RA *S*s performed somewhat more poorly than did the R *S*s ($p = .06$) suggests that having an easy to learn answer available may have detracted from learning the rule.

After three weeks the rule-given *S*s were better able to solve new problems based on new rules than were the discovery *S*s. How was this possible? Unfortunately, it is impossible to tell for sure. An intensive analysis of the ten rules actually used would be required. All that can be said with certainty is that the discovery *S*s found it harder to learn the rules than did the rule-given Ss. The discovery Ss were forced to discover rules independently, those who were successful perhaps also acquiring skills appropriate for discovering new rules.[1] The rule given Ss were merely required to learn rules. The much higher level of learning in the rule-given groups may have more than compensated, so far as novel transfer was concerned, for not having practice in discovery available to them. Because of the complexities involved, the novel transfer test, in effect, served only as an exploratory probe. For this measure to be definitive -- that is, to demonstrate that giving rules is better than discovering rules -- it would be necessary to equate rule learning.

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[1] In the study by Scandura, Woodward, and Lee (1965), the "answer given" groups learned best.

A study by Gagné and Brown (1961), in fact, does indicate that self-discovery may increase ability to learn when original learning is equivalent and involves single-instance principles (i.e. simple associations). In that study, the Ss were presented with number series, such as 1, 2, 4, 8, 16, 32,..., and were either given or required to find rules (i.e. formulas), depending on the number of terms, for finding the sum of series beginning with the specified terms and continuing with the same pattern. After completing a preliminary program designed to acquaint the Ss with the concept of a number series and a number of terms relating to such series, three treatments were given. Then, after exposure to a series, one group (RE) was told the correct rule and presented with examples, another group (GD) was given directions as to how to find such a rule, and a third group (D) was told to find a rule and was given some hints as to what the rule might be. Since the error rate on all programs was reasonably low, it might be assumed that the three groups learned the desired rules to about the same degree. That is, after completing one of the programs each S was probably able to state the correct rule when shown each of the series taught. The learning could have involved simple associations between specific series and formulas, but, from the description of the programs given, more likely was of a conceptual type involving series types.[1] The GD group, and to a somewhat lesser extent the D group, had an opportunity to discover a stimulus processing rule; a rule which may have told them how to go about determining a formula for finding the sum of a series shown for the first time. The results indicate clearly that when original learning, in this case four series-formula pairs, is equated, discovery enhances the ability to learn. Although it was not possible for Gagné and Brown (1961) to specify exactly what enhanced this ability, the present discussion suggests that the answer will be based on a logical, rather than behavioral, analysis. Once specified, of course, it might be possible to more efficiently enhance "discovery" potential by stating a general principle.

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[1] The information given in the published description of the study makes it impossible to determine whether such an information processing skill could have obtained.

Gagné and Brown (1961) interpreted the treatment effects as being due to differences in what had been learned. This was a legitimate thing to do. Learning how to find a formula is clearly different from learning to use formulas. The difficulty, I think, it one of possible confusion. The formulas, themselves, were rules and such rules have formed the basis for most other studies comparing expository and discovery modes of instruction (e.g. Craig, 1956; Haselrud and Meyers, 1958; Kersh, 1958; Wittrock, 1963).

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[1] The point of the present analysis, of course, is that the determination of what is learned should be an explicit part of such research.

Perhaps one of the foremost arguments in favor of formulating research problems in the SFL is that one is forced to consider relationships between presented and test materials. The relationships become explicit in terms of the stimuli, responses, and principle involved Thus, when the principle is a decoding principle (e.g. Wittrock, 1963), the stimuli are enciphered sentences, and the responses are deciphered sentences. When the responses are formulas, as in the Gagné and Brown (1961) study, the principle involved is some unspecified higher order rule for determining formulas for finding the sums of series. The stimuli, of course, are the series themselves.

Reversal and Nonreversal Shifts.- Figure 6 characterizes reversal and nonreversal shifts. The stimuli shown all have two, two-valued dimensions. On the first discrimination, size is the relevant dimension; large and small characterize two concepts, each with a distinct response, + or -. Color is irrelevant. After S can reliably make the first discrimination, a second discrimination problem is presented. A reversal shift involves exchanging the two responses, large going with - and small with +. A nonreversal shift involves reacting to the color dimension, black objects going with + and white objects going with -.

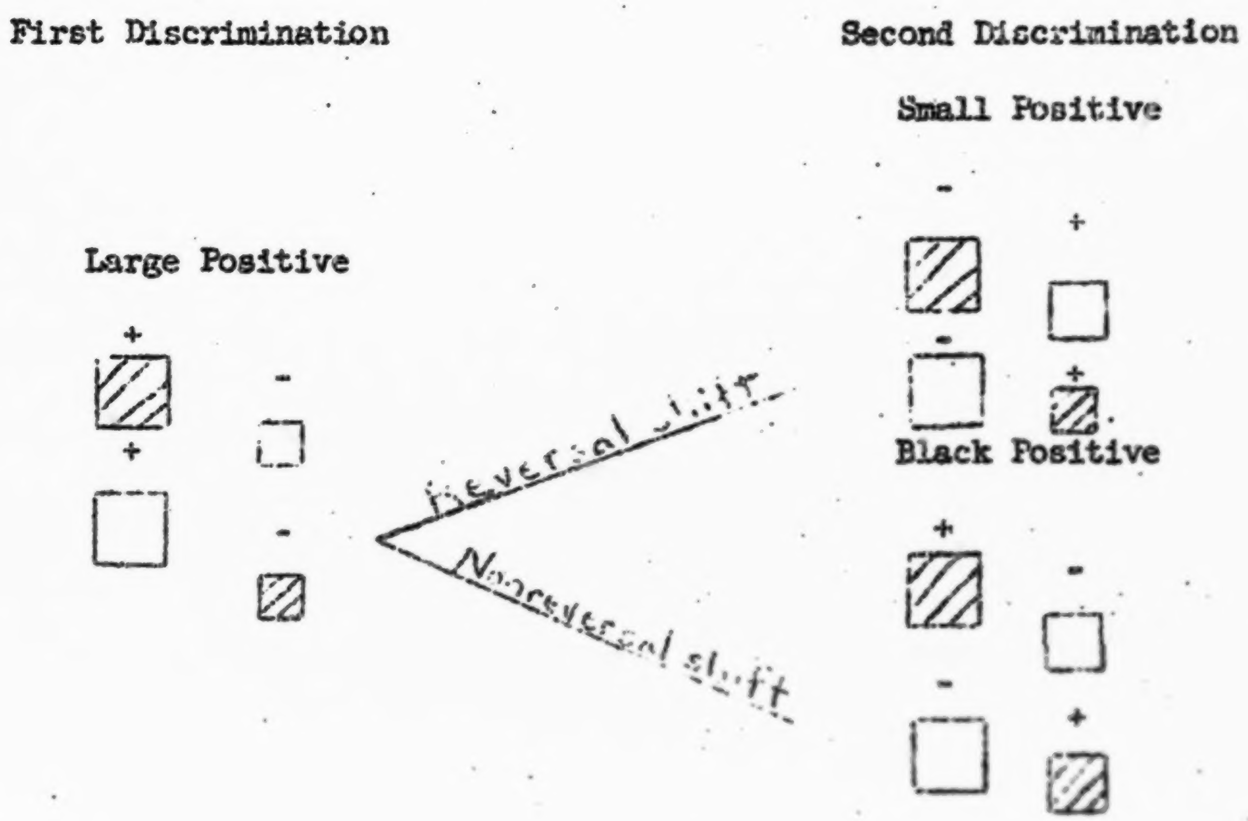
First Discrimination
Second Discrimination
Small Positive
Large Positive
Black Positive
Reversal shift
Nonreversal shift
+
-

Fig. 6. Examples of a reversal and nonreversal shift.

Pre-verbal children (Kendler, Kendler and Wells, 1960) and animals (Kelleher, 1956) find nonreversal shifts easier to make than reversal shifts whereas older more verbal Ss, find reversal shifts easier (Buss, 1956; Harrow & Friedman, 1958; Isaacs and Duncan, 1962; Kendler and D'Amato, 1955). In a study with kindergarten Ss, Kendler and Kendler (1959) found that fast learners, like verbal Ss, were better able to make a reversal shift whereas slow learners, like preverbal Ss, were better able to make a nonreversal shift. It was suggested that the fast Ss approached the experimental task with verbal labels for the correct stimulus already strongly attached, the verbal label serving as a mediating link in a two-stage S-R paradigm. The learning of the slower learning and presumably preverbal Ss was assumed to involve a single-stage paradigm. Kendler and Kendler (1962) explained the relative ease of reversal and nonreversal shifts in terms of the number of S-R associations that need to be changed.

This interpretation, however, leaves unanswered the question of whether the stated results were due to a progressive improvement in all children with age or to a larger proportion of faster learning children having whatever characteristic it is that makes reversal shifts easier.

Reformulating the reversal-nonreversal problem in the SFL provides a basis for answering this question. Perhaps the relative ease of shift is dependent on "what is learned" on the first discrimination -- two concepts (e.g. "If large, then +" and "If small, then -,") or four discrete pairs. If concepts are learned, reversal shifts should be easier since this merely involves learning two new responses. The principle identifying cues (e.g.

large and small) remain constant. A nonreversal shift would involve learning either two new concepts or four new discrete pairs. On the other hand, if four discrete pairs had originally been learned, a reversal shift would involve learning four new responses, whereas a nonreversal shift would involve learning only two.

Of course, this interpretation is analogous to that presented by S-R theorists (e.g. Goss, 1961; Kendler and Kendler, 1962). It is the assessment methodology which provides the means for determining whether learning type is related to relative ease of shift.

In order to assess what is learned on the first discrimination, it is necessary to employ dimensions, such as color (e.g. black, shaded, white) and shape (e.g. circle, square, triangle), which have more than two easily discriminated values. This procedure makes it possible to use two values of each dimension on the first discrimination problem leaving the others for assessment purposes. Thus, for example, the four training stimuli might be either black or shaded and a circle or a square. If reinforcement is given according to "color," the assessment procedure might involve presenting a new discrimination problem in which the two stimuli have the two color attributes, used during training (e.g. black and shaded), and the shape attribute not so used (e.g. triangle). Choosing, as positive, the object having the same color as the positive training stimuli, would be indicative of concept learning (on the training task) were it not for the high probability ($\frac{1}{2}$) of choosing this object by chance. Assessment "certainty," of course, can be improved by using more test discrimination tasks. This is made possible by increasing the number of values per dimension. In order to minimize "strategy shifts" during the assessment procedure reinforcement should be given at each choice point no matter what the response.

There is another way of reformulating reversal and nonreversal shifts in SFL terms. In an important sense, the problems posed are different from any previously encountered in this paper. For one thing the stimuli are pairs of objects. In addition, the response determining attributes may be relationships. Learning to discriminate between the object pairs shown in Figure 6, for example, can be accomplished by learning a principle of the form, "If shown two objects, then choose the larger one." Such a discrimination can also be accomplished by learning each principle in some nonredundant partition of this principle -- e.g. "If shown two objects and one is large and black (white), then select large black (white) object."[1]

In spite of the apparent differences involved, this formulation also leads to a similar analysis of this shift problem. If a single relational principle is learned on the first discrimination, reversal shifts should be easier since this merely involves learning a new response -- pick the smaller one. The principle identifying dimension, different sizes, remains constant. A nonreversal shift would involve learning either a new general principle or at least two new subprinciples in a partition. On the other hand, if the original partition had been learned, a reversal shift would involve learning at least two new subprinciples, whereas a nonreversal shift would involve learning only one.

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[1] It should be emphasized that S need not be aware of verbal labels in order to use a principle. Here, the labels serve primarily a definitional function.

It is also worth noting that discrimination learning may be viewed in terms of relational principles. Although not treated here, there is reason to believe that transposition phenomena (cf. Hebert & Krantz, 1965) may also be reformulated in terms of principles.

Unfortunately, there is no relevant data. Because the hypothesis and questions raised have obvious implications for studying interactions between developmental and learning problems, research on this question is urged. The results also would provide additional information concerning the appropriateness of the SFL and the related assessment methodology.

Conservation versus Non-Conservation.- Questions relating to the conservation (i.e. invarience) of certain concepts, such as amount, height, and number (e.g. Flavell, 1963), comprise another problem area in cognitive development that can be reformulated in SFL terms.

Consider a procedure that is frequently used to determine whether a child has learned to conserve "amount." The E shows the child two balls of clay, both of the same size. S is asked whether each ball, in turn, contains more clay than the other or whether they contain the same amount. Invariably the normal child says they are the same. E then rolls one of the balls into the shape of a sausage as shown in display two of Figure 7 and then asks the same question. If the child consistently says that they are the same, he is said to conserve amount. If not, he is a nonconserver.

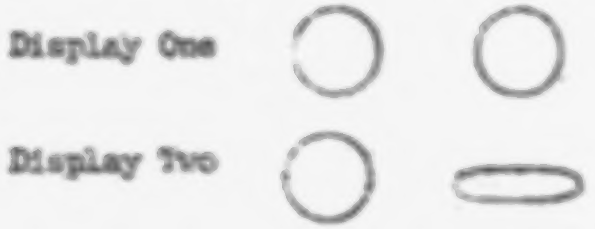

Figure 7. Two displays designed to determine the conservation of amount.

According to Ed Palmer,[1] the nonconserver apparently responds on the basis of length. A SFL formulation of this problem suggests that this is indeed the case. In order to properly formulate this question, however, all of the important communalities between the two objects in each pair must be identified. Not only do the objects in display one have the same amount, but they have the same height, length, width, and shape.[1]

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[1] personal communication

The child could respond on the basis of any one or more of these attributes. The word "amount," or whatever word is used by E, can have an entirely different significance for the child. The problem, of course, is complicated further because these dimensions are not all independent.

For the sake of argument, assume that the child responds according to either amount or length, but not both. In display one, the values of both dimensions are identical. Thus, a correct response could signify the operation of either of the two principles, "If shown two clay objects, then the response depends on the relative amount (length)."[2] Display two might be presented to determine which of these two principles is operating. If the child now says the sausage contains more clay, the operating principle probably involved length. If he says "the same," amount was likely the determining factor. It is worth noting that both of these two responses also eliminate height from consideration as a basis for responding. The former does not, however, eliminate shape.

One further point needs to be made. In developmental situations of the sort described, conservation of amount is assumed to have been acquired or not acquired prior to the experiment. If S does not demonstrate conservation, it is not a question of choice but of necessity. The principle of length may be operating, rather than that of amount, not just because S prefers it, but because he has not acquired the concept of amount.

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[1] It also could conceivably be of interest to distinguish between volume, cross-sectional area, and weight.

[2] These two principles, of course, refer to the generic basis for the child's response and have nothing to do directly with the principle E intends.

This may not necessarily be true. The child may have previously acquired both the concepts of amount and length in a generic, nonverbal, sense. All that can be said with confidence is that the phrase, "which contains more clay," may be interpreted by the child in one or the other of two ways.

CONCLUDING REMARKS

The basic point of this paper is that the principle can be made to serve as the basic behavioral unit. I have attempted to show that doing so has several important advantages over using the S-R association. An analysis of learning in terms of the principle allows one to consider situations of far greater complexity than anything that has been attempted via the S-R mediation language. When viewed in terms of principles, the critical dimensions of complex stimuli, in applied settings (e.g., arithmetic series), are often not nearly so difficult to determine as they are with nonsense syllables or words. In this regard, I would like to emphasize that SFL symbolism, like S-R schemas, provides a means for dealing with particularly complex situations in which ordinary discourse becomes ambiguous. It should also be noted that imposing mathematical constraints on functions has provided a natural means for classifying learning types -- some of which conform to classically made distinctions (e.g., association versus simple chain) and some of which do not (e.g., two representations of the "many-1" S-R pattern).

Perhaps more important, making central the notion of the principle leads one to ask different questions. Such important practical matters as generality, what is learned, and abstractness, for example, have lent themselves to rather close scrutiny. It is anticipated that future research may help unlock many of the behavioral mysteries that have long been attached to these ideas.

The present discussion of nominal and functional stimuli indicates the necessity of distinguishing between principle identifying cues and response determining attributes. Although a natural generalization, when the principle becomes central, this distinction is not made in the S-R language. Perhaps the primary reason that such a distinction, to my knowledge, has not been made before is that in the classical learning types studied by S-R experimentalists (e.g., association, simple chain, concept), the identifying and determining cues are identical. The functional elements in common between original and transfer stimuli serve both to identify similarities (and differences) and determine responses. The dichotomy involved in principle learning provides a direct basis for separating the psychological processes of discriminating between stimuli and of responding to a stimulus.

Viewing behavior in terms of principles also makes possible a feasible formulation of the vexing problem of "What is learned." After assumptions have been made regarding the underlying stimulus values and dimensions, learning can be defined in terms of observable test stimuli and responses. It would appear that such an assessment methodology may provide a necessary basis for an increasingly called for multi-stage approach to complex learning and teaching. Under these conditions such research may be expected to make more rapid progress.

In spite of the rather far reaching basic assumption made in this paper, no attempt has been made to even outline a theory based on the notion of a principle. It is certainly to be hoped that no major changes need to be made regarding such sacred principles as contiguity and reinforcement.

Since they have guided much of the preceding development, I feel obligated to make certain conjectures as to the possible nature of some additional theoretical assumptions. First, the response to a new stimulus is determined uniquely by the operating principle. Regardless of how principles are acquired, behavior, according to the present discussion, depends on the selection and use of some principle. Second, principles are assumed to continue in operation unless disrupted by the conflicting influence of new input and/or feedback. According to this postulate, responses to new stimuli remain under the control of a particular principle unless either the stimuli do not correspond to the principle or feedback otherwise indicates that the rules have changed. This notion has much in common with the mathematics modeler view (e.g., Bower and Trabasso, 1964; Levine, 1964; Restle, 1962) that S changes hypotheses only when given negative reinforcement (however, see Suppes and Schlag-Rey, 1965).

Clearly, these two postulates do not, in themselves, constitute the basis for any theory; existing principles are modified and new principles are acquired. They are much like Newton's laws of reaction and inertia without an $F=ma$ to tie them together with the dynamics of change. Although some guesses could be made as to the nature of such a postulate, it would be inappropriate for me to speculate further at this point.

One final comment seems in order. When carried to the extreme, it becomes clear that the principle, rather than the association, is the basic behavioral unit. No two stimuli or responses are exactly the same. When we speak of a discrete association we are, in actuality, referring to a set of S,R pairs in which the physical stimuli and responses are almost identical.

If the preceding analyses bear any weight at all, it would appear that the future progress of research on complex learning and instruction will be largely dependent on the ability to deal simultaneously with structural and behavioral variables in both a precise and simple way. The SPL is presented as one alternative.

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USING THE PRINCIPLE TO FORMULATE RESEARCH ON MEANINGFUL LEARNING*

(A Set-Function Language)

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The search for a suitable scientific language in psychology has had a long history. Unfortunately, as with theories, there is no *a priori* basis for deciding between alternatives. Which will prove most useful can be determined only after a period of use. Nonetheless, certain characteristics appear desirable. One of these is precision. The primary requirement, however, is that the language accurately represent the important characteristics of the phenomena in question. Without such fidelity the language can have no real value--it is the *sine qua non*.

In order to construct a precise descriptive language, which adequately reflects meaningful learning, a basic behavior unit must be selected. The history of science has shown that the hypothesis-generating and predictive value of any theory or scientific language is determined in large part by the appropriateness of its basic building blocks.

Many theorists have been primarily concerned with extending S-R formulations to account for complex phenomena (e.g., Berlyne, 1965; Kendler & Kendler, 1962; Maltzman, 1956; Goss, 1961; Osgood, 1953; Staats & Staats, 1964). Although it has been repeatedly emphasized that the S-R approach is simply a means of working, of baring essentials, the neo-associationist implicitly believes that the association provides the *most*

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precise and efficient unit with which to describe behavior.

The fundamental, and perhaps most questionable, assumption underlying neoassociationism is that mediating links in an S-R chain have the same properties as overt S-R associations (Berlyne, 1965, 17-19). In view of the success achieved in viewing animal and simple human learning in terms of associations, parsimony would seem to call for such a principle. Yet, practice has shown that mediation interpretations become increasingly cumbersome and less precise as situations become more complex.[1] More important, Anderson (1964) and Fodor (1965) have recently argued convincingly that multi-stage explanations only give the appearance of greater explanatory power. Single stage formulations can always be devised which are equivalent.

Largely for these reasons, other theorists and highly reputable writers (e.g., Ausubel, 1963; Bartlett, 1930, 1958; Dienes, 1963; Gagne, 1962, 1965; Mandler, 1962, 1965; Miller, Galanter, & Pribram, 1960; Piaget, as described in Flavell, 1963; Polya, 1962, 1965; Newell, Shaw & Simon, 1958) feel that the S-R language does not capture the essence of meaningful learning. Typically, they find the idea of an association or network of associations to be incapable of reflecting all that a human does when confronted with a problem situation. Constructs are needed to enable Ss to think (Mandler, 1965, 325). Thus, Bartlett (1930, 1958) speaks of organization and rules, Gagne (1962, 1965) of knowledge and principles (and learning sets), Mandler (1962, 1965) of structure, Miller, Galanter, & Pribram (1960) of TOTE units (and heuristics), Piaget (Flavell, 1963) of schemas, Polya (1962, 1965) and Newell, Shaw, and Simon (1958) of heuristics, and Tolman (see Hilgard, 1956, 191) of cognitive maps and sign-significate relations.

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[1] The present practice of repeatedly extending associationistic schemas to account for new facts, particularly as regards meaningful learning, is highly reminiscent of pre-Copernican astronomy. At that time too, emerging facts were incorporated into what we know now to be an unnecessarily complex (geocentric) theory.

Several of these writers (e.g., Miller et al, 1960; Polya, 1962, 1965; Newell et al, 1958) have dealt with problem solving in its full complexity. Emphasizing the role of heuristics--broadly applicable modi operandi--they have been either of the opinion that problem solving should be treated as an art (Polya, 1962, 1965) or that the computer (Miller et al, 1960; Newell et al, 1958) provides the only really effective means for dealing with the complexities involved. The former view, of course, is antithetical to science. Computers, on the other hand, although they provide a valuable tool and possibly a viable model, do not alleviate the scientist of the responsibility for identifying the basic behavioral units and stripping theory of nonessentials. Computer simulation, due to the technical complexities and practical problems involved, may be as much a hindrance as a help in theory construction.

Others (e.g., Ausubel, 1963; Bartlett, 1930; 1958; Dienes, 1963; Piaget, as described in Flavell, 1963) have also offered appealing analyses of meaningful learning and problem solving, but they have been forced to gloss over many subtleties. There has been no sufficiently precise language available for formulating their ideas. Although Piaget has made considerable use of logic in his theoretical work, it has served primarily to describe internal capabilities. In tieing these capabilities to observables, Piaget has simply used the French language, sometimes in rather abstruse fashion.

In short, the choice, to date, has been between a precise, but seemingly inappropriate S-R language, and presumably more relevant cognitive formulations which leave much to be desired in so far as scientific cohesiveness and rigor is concerned.

Gagne's (1962, 355) statement to the effect that knowledge allows one to "perform successfully on an entire class of specific tasks, rather than simply on one member of the class" is indicative of the fundamental difference between meaningful and rote learning. Knowing how to add means that the learner is able to give the correct response to any addition problem, not just one. While not necessarily impossible for the neo-associationist to explain how Ss can give new responses to new stimuli, it does present difficulties.[2] In fact, the explanations offered (e.g., Berlyne, 1965, 168-171) are not really explanations but independent postulates.

Although Gagne (1964, 1965) has taken great pains to indicate how higher forms of learning, such as the principle, depend in turn on lower forms, such as the association, there remains the possibility that a higher form may be basic while the lower forms are simply special cases. Tracy Kendler (1964) has alluded to this possibility when, in reviewing Gagne's (1964) paper, she suggested that new properties may emerge at the principle level. Gagne's (1964) original representation of the principle did not use the S-R language. He (Gagne, 1965) has since attempted to reconcile this difficulty, by viewing a principle as a chain of concepts. The analogy, however, is somewhat strained and in no way debases the suggested alternative.

Principles appear to be involved in every act. We determine the sum of a number series according to some well learned rule or plan. We open the door when someone knocks, except possibly when working on a manuscript. Even as experimental subjects, we spew as many associates to a stimulus as we can according to some plan of operation--a plan frequently introduced by the experimenter.

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[2] See the discussions below concerning stimulus-response generalization.

According to Webster a principle may be defined as follows: "an underlying faculty or endowment; a governing law of conduct: an opinion, attitude, or belief that exercises a directing influence on life and behavior, laws or facts of nature underlying the working of an artificial device (or human)."[3] In short, the term principle can be used to refer to that which underlies behavior.

The purpose of this paper is to describe a precise (mathematical) set-function language (SFL), based on the notion of a principle, which appears to capture much of what has been proposed as central to meaningful learning. The paper has been organized as follows. First, the basic rationale is described and the principle is given a precise mathematical characterization, largely in terms of sets and functions. Second, the S-R and SFL languages are contrasted. Third, empirical research, which was based on a preliminary formulation of this language and which has aided in its development, is described. Fourth, problems concerning reception and discovery learning, reversal and nonreversal shifts, Piagetian conservation tasks, and symbolic and concrete learning, are reformulated in the SFL. Fifth, theoretical direction is given. Finally, some concluding remarks are made.

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[3] The parentheses are mine.

THE PRINCIPLE--BASIC UNIT IN THE SET-FUNCTION LANGUAGE (SFL)

To provide motivation for the mathematical characterization given below, consider the following situations: (1) a young child learns to say "16" when he is presented with the nominal stimulus "1 + 3 + 5 + 7"--i.e., he learns to say the sum of the series represented (although he may not know what a sum is), (2) an algebra student learns to give the sum "16" when shown any one of several representations of the series 1 + 3 + 5 + 7 (e.g., "1 + 3 + 5 + 7," "$\sum_{m=0}^{3} (1+2m)$," etc.), and (3) a college student learns a rule for finding the sum of any arithmetic series.

Before discussing differences, consider first what these situations have in common. In each case, learning may take place in either one of two ways. The to-be-learned material can be presented directly, as in reception learning, or the material may be learned by discovery. In the three situations described, the following statements might serve to promote reception learning: (1) "If shown '1 + 3 + 5 + 7,' then say '16', and (2) "If shown any representation of the series 1 + 3 + 5 + 7, then say '16', and (3) "If shown (any representation of) any arithmetic series, then say the numeral corresponding to $\frac{(A + L)}{2}N$ where A is the first, L the last, and N the number of terms in the series." In acquiring such abilities by discovery, the stimulus-response pairs involved may be presented, in turn, until the learner can correctly anticipate the next response to a new stimulus within the same class. To acquire an underlying principle, S must encode both the stimulus and the response and discover a common relationship between them. In situation one, of course, this relationship is simply a direct connection between the internalized stimulus and the internalized response.[4] In the other situations, the relationships are between the elements common to each representation of the related stimuli and those common to the responses. In situation two, the series, 1 + 3 + 5 + 7, is an abstract entity (property) common to each stimulus, the common response property is simply the internal representation of "16." In situation three, the common properties refer to series of the form a + (a+d) + (a+2d) + ... + (a + (n-1)d) and numerical sums, respectively.

The second characteristic common to these situations involves performance. Making appropriate use of knowledge, implies that an information processor is able to determine, not only what response to give to a class of stimuli, but also when to apply what he knows. To accomplish this the stimulus situation must be encoded in a manner appropriate to the context, the operation must be carried out, and the results must be made observable.

The general context within which selected ~~a~~ plan is presumably provided either by directions or by internal stimulation of some sort.[5] Within a given context, however, stimulus properties determine which principle is appropriate.

Suppose the inputs of concern are general number series of the form $a_1 + a_2 + ... + a_n$. Within this context, the principle involved in situation one, would be applied whenever "1 + 3 + 5 + 7" appeared. In situation two, any representation of 1 + 3 + 5 + 7 would do. The principle, associated with situation three, would be applied to any arithmetic series, number series in which a common difference exists between adjacent terms. Furthermore, all of the common stimulus features need not be involved in determining the appropriate response to a given stimulus. In situation three, the first, last, and number of terms in the series would serve this purpose. In effect, encoding a stimulus may serve two distinct functions, to identify the appropriate principle and to determine responses.

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[4] In this situation, each presentation of the single pair involved would have to count as a different pair--in fact, they are different if only in presentation time.

[5] It is possible that directions and internal stimulation may be viewed as higher order principles of some sort, principles which narrow the range of subordinate principles which might be evoked in a given situation. This possibility is discussed to some extent in later sections, but detailed consideration is beyond the scope of this paper.

In many ways the problem of ascertaining what will be encoded is easier with meaningful than with nonsense materials. With meaningful materials, because we have some basis for knowing what prior knowledge will be brought to bear on the decoding task, one can frequently assume that inputs will be encoded in particular ways and that internal representations of responses can be made observable. For example, "5 + 2 x 3" will undoubtedly be interpreted by the reader as "5," "+," "2 x 3" and not "5 + 2," "x," "3." The reason for this phenomenon does not reside in the stimulus but is based on a previously learned convention. Similarly, all verbal Ss can presumably write (and say) the numeral "6" representing the (internalized) number 6.

The nature of the relationship between the underlying principles and the observable stimuli and responses also appears to be the same in all three situations. Whether the learning be by reception or by discovery, presenting test stimuli within and outside the scope of particular principles would appear sufficient to determine what is learned. Since more than one principle may lead to the same response, however, the problem is not quite that simple. Questions concerning assessment are discussed in more detail in a later section.

Although learning and performance are of an apparently similar nature in the three situations described, there are important differences. In situation one, the relevant attributes could be almost any combination of salient features in "1 + 3 + 5 + 7." This is also true of the combining rule selected. A likely possibility, however, would be a simple mapping (i.e., association) between what is encoded and the internal representation of 16. In situation two, the relevant attributes may be common to each representation of the series 1 + 3 + 5 + 7. If only number series are considered, both of the sets [1, 3, 5, 7] and [1 (first term), 7 (last term), 4 (number of terms), 2 (common difference)] would serve to characterize these communalities. That none of these properties (e.g., number of terms, 4) can be used to distinguish between the various possible series representations, suggests that these properties are to be associated with *sets of stimuli* and not the stimuli themselves.[6]

Either set of common attributes can be used to determine the appropriate sum response. The terms 1, 3, 5, and 7 can be added sequentially and the expression "($\frac{1 \text{ (first term)} + 7 \text{ (last term)}}{2}$) 4 (number of terms)" can be simplified. In effect, these situations are entirely analogous to learning concepts which involve *both* attribute identification and rule learning (Haygood & Bourne, 1965).

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[6] In mathematics, properties are often equated with sets of elements which have these properties. These elements, in turn, are sets. Thus, the number two is defined as that collection of sets each having two elements (e.g., oranges, apples, things). The algebraic symbol, N, is a still higher order (more abstract) property and refers to a higher order collection which contains the various collections to which the integers refer. In the same way, the series, 1 + 3 + 5 + 7, also refers to a class of stimuli--those, for example, corresponding to the various ways of representing 1 + 3 + 5 + 7 (in numerals or words; with pencil, pen, typewriter, etc.). Even a property such as "red" is equated with a set of elements rather than attributed to a stimulus object. In most situations, of course, this distinction between a property of an object and the property itself (which is equated with a class) can be ignored. But there are other circumstances where the distinction is important (e.g., whenever the psychologist wishes to distinguish between an (overt) stimulus property and an internal mediating response (property) itself. In what follows, I shall ignore this distinction where the intended meaning is clear.

In situation three, the relevant attributes are common to all arithmetic series. Again, the communalities can be characterized by at least two sets of properties, $a_1, \ldots, a_1+(N-1)D=a_n$ and $a_1 = A$, $a_n = L$, N, D. The difference between this and situation two, however, is more fundamental than simply involving a larger class of related stimuli. *The responses also vary.* The internal representation of the responses also is a property relating to a whole class of (sum) responses. The introduction of a response dimension, in addition to the stimulus dimensions existing in situation two, results in stimulus properties of a higher order in situation three in the following sense. These latter properties refer to collections (sets) of sets of arithmetic series, each of which, in turn, has a variety of symbolic representations. The first term, A, for example, is a property of the class of those sets in which the arithmetic series have a common first term. A single combining rule (e.g., $\left(\frac{A + L}{2}\right)N$), by which the sum of any arithmetic series may be obtained, could provide a basis for responding as indicated in situation three. This rule involves all that is common to a class of rules for determining particular sums (e.g., $\left(\frac{1 + 7}{2}\right)4$, $\left(\frac{3 + 9}{2}\right)15$, etc.). In short, A, L, N, and $\left(\frac{A + L}{2}\right)N$ are of a higher order than 1, 7, 4, and $\left(\frac{1 + 7}{2}\right)4$, respectively. They are said to be more abstract.[7]

In summary, learning may occur by either the direct interpretation of given information, such as a statement of principle, or by discovery, the abstraction of a relationship common to a whole class of stimuli and responses. In each case, the basic behavioral unit is an internalized plan of action whereby an information processor--human, rat, or computer--may determine which responses to make to which stimuli. A principle is not directly observable and its presence can only be inferred from the behavior resulting from operating on particular stimuli. All observable inputs pertaining to human learning are not designed to elicit responses. Inputs may also serve to promote learning and to indicate the context in which principles are used. The presentations both of a statement of principle and of an S-R pair in paired-associate learning provide illustrations of the former type of input; directions provide an illustration of the latter.

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[7] Notice that it is easier to detect a particular stimulus property when presented with a lower order symbol, such as "1," than one of higher order, such as "A." Being able to determine any value of A, for example, necessarily implies that "1," "2," etc., can *all* be determined. Higher order rules and properties may be fundamental to most human behavior--particularly with respect to such subject matters as mathematics.

In spite of these communalities, the nature of what is learned may differ. The learning may represent a simple relationship between the internal representation of a single stimulus and a single response; a relationship between representations (set properties) of a set of stimuli and a single response; or a relationship between representations of a set of stimuli and a set of responses. In effect, two types of generality may be involved. The principle may refer to one stimulus and one response, a class of stimuli and one response, or a class of stimuli and a class of responses. In the latter situation, a second level of abstraction is imposed on the stimulus properties.

Principles, denotations, and statements can be given more precise mathematical definitions. In order to distinguish one principle from another the following must be specified: (1) the set of identifying properties (I), which make it possible to determine when the principle is to be applied, (2) the set of response determining properties (D), from which the response is derived, (3) the set of response properties (R), and (4) the combining rule, mapping, or operation (O) by which R is derived uniquely from D. In short, a principle may be characterized as an ordered four-tuple, (I, D, R, O) where I, D, R, and O are defined as above.[8]

The denotation of a principle consists simply of the set of corresponding instances, the observable S-R pairs. This set may be denoted: $[(S_i, R_i) | i = 1, 2, \ldots,$ and (S_k, R_a) and (S_k, R_b) implies $R_a = R_b]$. This definition implies that there can be only one response paired with each stimulus--the denotation of a principle simply being a function[9] in which the domain is the set of all observable stimuli associated with the principle and the range includes all of the observable responses.

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[8] A principle may also be characterized in terms of description spaces (Hunt, 1962). Essentially, a description space is simply a product space in which the one dimensional component spaces are stimulus dimensions. Each dimension is partitioned into categories. For example, consider a description space consisting of the three dimensions, size, color, and shape, with the values (categories) large and small, black, white, and green, and triangle, respectively. Any stimulus object with some combination of these values may be placed in one of the resulting six (2 x 3 x 1) categories.

A derived description space is simply another description space derived from the first by mapping points (descriptions) in the first space into a second space with at most the same number of dimensions and values. The derived dimensions and values may be identical or different from those in the original space. The set of properties and dimensions, D, corresponds to a description space; R corresponds to a derived space; and O corresponds to the mapping from one to the other. Notice also that identifying the properties in I may depend on "operating" on lower order stimulus characteristics. Thus, a series is arithmetic only if certain properties hold--e.g., the difference between adjacent terms is a constant. Although it may ultimately prove useful to treat I as a derived space, along with R, this does not appear to be necessary in most situations and will not be considered further in this paper.

This characterization in terms of description spaces has certain advantages; it is precise, can be represented graphically, and emphasizes the distinctions and relationships between values and dimensions. It has the disadvantage, however, of requiring somewhat more mathematical background than is necessary for present purposes. For this reason and because of space limitations, no further use of these notions is made in what follows.

[9] Mathematically, a function is a set of ordered pairs, (x, y), in which each value of the first variable, x, is paired with only one value of the second, y.

A statement must faithfully reflect the corresponding principle. To provide an adequate symbolic (observable) representation, it should make reference to all that characterizes the principle. This may be accomplished by a statement of the form, If I', then $R' = O'(D')$," where I', D', R', and O' are symbolic representations for I, D, R, and O, respectively.[10] Primes are not used in what follows, except where necessary for clarity. Although statements can be given in different forms and such differences may be important, especially at early developmental levels, subsequent discussions are limited primarily to this form.

To prove useful, a new scientific language must in some substantive way represent an improvement over existing formulations. First, it may provide a more precise and parsimonious basis for describing important phenomena than do existing languages. Second, it may lead one to ask new and important questions and help make it possible to reformulate existing questions in more researchable form. If a new language should do only as well, and no better in both of these respects, one can seriously question the advisability of changing the scientist's frame of reference. This is particularly true in the present case where the approach taken does not simply represent a superficial departure or elaboration of an existing, and apparently useful, language.

In the following sections, SFL and neo-associationistic (S-R) formulations are compared and research suggested by the SFL is described.

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[10] The role symbolism plays in learning and interpreting principles is discussed in a later section.

COMPARISON OF SFL AND S-R LANGUAGES

Although it also is tied inextricably to observables, and is therefore behavioristic, the SFL is not "associationistic," nor even "neo-associationistic, in the traditional sense of these terms (e.g., Berlyne, 1965). In the SFL, a clear distinction has been made between observables (e.g., denotations and statements) which induce or reflect internal motivations or *modi operandi* and those inputs on which such internalizations operate. To be sure, the organism is thought to operate on the stimuli; the stimuli do not operate on the organism (although, of course, they provide the *occasion* for making particular responses).

In order to consider meaningful materials and the important role of internal stimulation (motivation), S-R analysts have chosen to extend empirically determined properties of overt stimuli and responses to mediating processes. In effect, the constructs are assumed to have the same properties as the observable phenomena they are seeking to explain. It may be this sort of circularity that has led some writers (e.g., Anderson, 1964; Fodor, 1965) to severely question the explanatory power of intervening stimuli and responses.

In short, the crucial difference between SFL and S-R approaches appears to be the nature of the construct used to tie observables together. In the former, the association is the fundamental building block; in the latter, the principle is basic.

Principle versus Mediating S-R Link. - Fundamentally, both principles and mediating S-R links represent a learned connection between one set of observables (stimuli) and another (responses). Mediating links, however, are often combined into chains of indeterminate length and various interconnections between different chains are often postulated for explanatory purposes. In the SFL, although a counterpart to chaining exists, only one-stage connections are necessary.

Of perhaps more concern, mediating S-R links refer to only one type of connection--associations between stimuli or mediating stimuli and mediating responses or responses. The combining operation, O, referred to in the SFL characterization of a principle, is more general and refers to a mapping between mediating responses (e.g., D) and their stimulus properties (e.g., R). The almost exclusive use of the association in representing simple learning, both animal and human, has not greatly hindered American behaviorists since general rules or mappings have played no important part in their research. Even research on concept learning has been largely limited to problems which have lent themselves to S-R mediation arguments. It has only been very recently that the rule learning aspects of concept learning, for which no ready associationistic representation is available, have been dealt with explicitly (Haygood & Bourne, 1965). These authors independently recognized that concept learning involved both (stimulus) attribute identification and rule learning. The present argument, of course, is that rule learning plays an even more important role in the learning of principles which relate to a class of responses (as well as a class of stimuli).

Another major difference between the SFL and S-R languages involves the internal representations of stimuli. In the SFL, certain aspects of the stimulus may serve to cue the principle and other aspects may determine the response. With respect to the principle, "If arithmetic (i.e., there exists a common difference), then sum equals $\frac{(A + L)}{2}N$," for example, the identifying (common difference) and response determining (A, L, and N) characteristics of arithmetic series are distinct. In the S-R language no such distinction is made; the same functional stimulus serves both functions.

Distinguishing between the roles played by stimulus properties forces stimulus, response, and stimulus-response generalization, as originally postulated by Hull (1943), to be viewed in new light. Generalization phenomena are provided a rigorous basis in terms of "what is learned." The properties in I determine the variety of stimuli to which a principle is applied. The fewer characteristics required to identify a principle, the more widely applicable the corresponding principle will be. This follows directly from the fact that the number of eligible stimuli varies inversely with the number of required properties (i.e., restrictions) imposed. The properties in D, on the other hand, determine what the corresponding responses are. The number of responses involved depends on the number of response determining dimensions (n) and the number of distinguishable values of each. Each n-tuple of determining cues (one for each dimension) corresponds to one response. For example, the properties (i.e., dimensions) "color" and "shape" (in D) would result in a larger number of responses than "color" alone. The more different stimulus properties referred to by the encoded dimensions, the larger the denotation of the principle.

It is important to note that D and O may make it possible to determine responses (that may not be correct) to stimuli outside the scope of a given principle. For example, N^2 leads to the incorrect "sum" 9 to the series 2 + 4 + 6. In an analogous fashion the properties in I may overly restrict the set of stimuli to which a rule is applied. The SFL characterization of a principle provides, in effect, a basis for independently manipulating those cues (I) responsible for generalization (and discrimination) on the stimulus side and those cues (D) responsible for generalization on the response side.[11]

As we shall see below, the stimulus identifying and response determining cues are identical with respect to the learning of associations and concepts, which have provided the subject matter for the vast majority of experimental studies. Since the S-R language makes no explicit provision for dealing with the phenomena, it is not surprising~~,~~ that as Berlyne (1965, 169) recently points out, "Stimulus-response generalization, in which the formation of an association between S_1 and R_a results in the formation of an association between S_2 and R_b, has, ..., scarcely been investigated at all." This is hardly a desirable state of affairs if principles are as critically involved in meaningful learning as has been proposed (e.g., Gagne, 1962, 1965; Scandura, 1967, 1967).

While on the topic of generalization, I might add that the present concern with discrete variables is not as limiting as might be expected. First of all, the examples chosen suggest that the critical values of meaningful stimuli can often be readily distinguished. The numeral "6" represents the number six and not five. Problems involving "just noticeable differences (jnd)," as in psycho-physics, are rarely of primary concern with subject matters like mathematics and probably others as well. In some ways, the analysis of actual subject matters may be less difficult and more precise than that of nonsense syllables. Second, the generalization gradient, almost always noted when training and transfer stimuli differ along a continuous dimension, may be an artifact due to averaging over different jnd. If this latter supposition proves sound, it could have important implications for the implicitly assumed probabilistic basis for behavior. It may be that probability enters the situation largely because the properties in the set I have not been identified.

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[11] The third type of generalization postulated by Hull (1943), response generalization, may in reality be nothing more than stimulus-response generalization. Although maintaining Hull's (1943) original trichotomy, Berlyne, (1965), admits that differences between response generalization and stimulus-response generalization may be primarily a matter of convention. In view of the preceding discussion, in which counterparts for stimulus and stimulus-response, but not response, generalization were identified, it may be advisable to map this theoretical trichotomy into a dichotomy. While not denying the fact of response generalization, I am questioning its theoretical necessity.

Data reported in the next section suggest that there is no decrement in performance on training and transfer stimuli (within the scope of a principle) when the stimulus cues are discrete and readily determinable.

Classification of learning types. - Current interest in taxonomy development has been intense (e.g., Melton, 1964; Stolurow, 1964). More important, there has been a concerted effort to uncover basic similarities between what have heretofore been considered separate categories (e.g., Fitts, 1964; Gagne, 1964, 1965). The emphasis has been towards genotypic, rather than phenotypic, bases.

Perhaps the most encompassing classification scheme of this sort, which is based primarily on S-R terminology, is one proposed recently by Gagne (1964, 1965). This taxonomy provides a natural, and I think fundamental, basis for comparing the S-R and SFL languages. The association is the basic building block in Gagne's formulation; most of the higher forms of learning represent complications of this unit. When reformulated in the SFL, the principle becomes basic and the simpler forms special cases.

Gagne (1964, 1965) identifies eight types of learning: (1) signal learning--the establishment of a conditioned response which is general, diffuse, and emotional, and not under voluntary control, to some signal, (2) S-R learning--making very precise movements, under voluntary control, to very specific stimuli, (3) chaining--connecting together in a sequence two (or more) previously learned S-R pairs, (4) verbal association--a subvariety of chaining in which verbal stimuli and responses are involved, (5) multiple discrimination--learning a set of distinct chains which are free of interference, (6) concept learning--learning to respond to stimuli in terms of abstracted properties like "color," "shape," and "number," (7) principle learning--acquiring the "idea" involved in such propositions as "If A, then B" where A and B are concepts; a chain or relationship between concepts, internal representations (of concepts) rather than observables being linked, (8) problem solving--combining old principles so as to form new ones.

According to Gagne (1965, 30-31), these varieties of learning were determined in accordance with the conditions required to bring them about. Thus, for example, the preconditions for signal learning are the nearly simultaneous presentation of two forms of stimulation, UCS and CS. Those for principle learning are the prior learning of related concepts and the chaining of these concepts.

This scheme has the advantages of having a practically important base and of at least formally relating simple learning types with more complex forms like Gagne's principle (GP). (To minimize the possibility of confusion I shall use GP to refer to the sense in which Gagne has used the term principle). Still, there are important limitations. Gagne's original representation (1964) of principles and problem solving did not use the S-R language. In his later formulation (Gagne, 1965) he has attempted to show similarities by considering principles to be chains of concepts, but the analogy is somewhat strained. Concepts are not directly observable whereas the S-R links in a chain can supposedly be made so. Using chaining mechanisms in both situations may indicate similarities which are more apparent than real.

No such difficulties are encountered when the learning types are represented in the SFL with the counterpart of GP taken as basic. GP may be characterized (I, D, O, R) where the properties in I and those in D refer to stimulus dimensions (i.e., higher order properties), O refers to a class of operations (i.e., an abstract operation relating a set of stimuli and a set of responses), and R refers to a class of responses. The denotation of such a principle, symbolized $[(S_i, R_i) \mid i = 1, 2, \ldots$ and for all $S_k, (S_k, R_a)$ and (S_k, R_b) implies $R_a = R_b]$, consists of a set of S-R pairs with a variety of different responses and with each stimulus paired with only one response. The representation of a principle statement in the SFL, "If I, then R = O(D)," is of the same form, but more detailed than that, "If A, then B," used by Gagne (1964, 1965).

The other types of learning, identified by Gagne (1964, 1965), turn out to be either special cases or a composite of principles. Concepts are simply principles in which the properties in I = D refer to a class of stimuli, but are not dimensions, and those in R refer to a single response. Although not a necessary condition, O is simply a logical rule for combining relevant attributes in most studies of concept learning (e.g., Haygood & Bourne, 1965; Bruner, Goodnow, & Austin, 1956; Hunt, 1962). The denotation of a concept, symbolized $[(S_i, R_i) \mid R_i = R, i = 1, 2, \ldots]$, consists of a set of S-R pairs where R is a constant.[12]

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[12] All many-one S-R pairings are not concepts in this sense. They may also consist of a number of discrete S-R associations with only one response.

In order to represent simple S-R associations and sign learning,[13] the SFL formulation is restricted still further. The (properties in) I = D refer to a single stimulus, R to a single response, and O to a simple mapping (i.e., association) between D and R. In this case, the denotation is a set consisting of a single S-R pair.

Notice that in much association learning, the response learning phase (i.e., the connection between mediating stimuli and overt responses) is of considerable importance (e.g., Underwood & Schultz, 1960). Of primary concern in higher forms of learning are the identification of critical stimulus cues (i.e., I and D which correspond to mediating responses) and learning the combining rule. The internal R-overt response pairing, such as between numbers (internal) and numerals (overt), are typically well-learned prior to concept and principle learning. Nonetheless, the application of a learned principle (statement) involves the perception (or determination) of the relevant stimulus properties which, in turn, are transformed into internal response properties (which previously have been tied to observable responses). In those cases where *S* is unable to determine the critical stimulus properties, carry out the rule, and make the results of the rule observable, successful application would be impossible. Thus, for example, applying a principle, where $O \equiv \frac{(A + L)}{2}N$ and D = (A, L, N), to find the sum of an arithmetic series, involves determining the properties in D. The determination of N, however, has proved to be a non-trivial task for junior high school Ss (Scandura, Woodward, & Lee, 1965).

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[13] Sign learning can be viewed as a principle which might be stated, "If a light appears, then elicit a *worry* response (since a shock will follow)." Clearly an animal cannot learn a verbal statement of this sort, but he can operate according to some such plan. There seems to be no *a priori* reason why animal, as well as human, behavior cannot be viewed in terms of principles. On the other hand, there may be no important reason to do so since animals have no language system for transmitting principles with all that undoubtedly implies for complex forms of learning and behavior.

Not having acquired the necessary "perceptual" (or operation, response learning) skills required to apply a particular principle, however, does not necessarily imply that S cannot make use of a different principle having the same observable characteristics (denotation). It is often possible to start at a lower level--with more readily discernible (stimulus) properties.[14] The number of terms, N, in an arithmetic series, for example, can be derived from the formula $O' \equiv \frac{(L + C - A)}{C}$, where $D' = (A, L, C)$ and C is the common difference between adjacent terms, C presumably being easier to determine than N. In this case, the derived rule, $O'' \equiv \frac{(A+L)}{2} \frac{(L+C-A)}{C}$, where $D' = (A, L, C)$, can be viewed as a composite operation with $O''(D') \equiv O[O'(D')] \equiv O(D)$. In this formulation, the operation O', of course, must be viewed as mapping each value of A, L, and C in D' into the corresponding values of A, L, and N in D. A composite operation, of course, has the same properties as any other operation. Unless the responses corresponding to each constituent operation are actually observed, guessing the stages a learner went through would appear to serve no useful purpose.[15]

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[14] According to this view, determining characteristics and perceiving characteristics are one and the same process--the difference simply being one of degree. Thus, just as "16" can be derived from "1," "7," and "4," "4" can be derived by counting the numerals in "1 + 3 + 5 + 7"--the numerals being still more easily discernible (than "4").

In most experimental studies, (involving perception directly), the concern has been primarily with less abstract characteristics--typically physical properties of the stimulus. In such cases the derivation rules are probably well learned. Still, all Ss at one time or other had to learn how to determine even physical characteristics. The new-born infant presumably is very limited in its ability to decode information. Perhaps, it is the few such abilities that it does have, such as withdrawal tendencies from aversive stimulation, which provide a basis for determining higher-order abilities.

The ability of the SFL to effectively represent multiple discrimination learning depends on the principle identifying properties. Presumably, a number of discrete S-R pairs become well-learned and interference free when the sets, I, of identifying properties, corresponding to the distinct pairs, are disjoint. The denotation of multiple-discrimination learning would consist of a number of distinct one-pair sets.

Notice that the principles involved in multiple-discrimination learning can be of a more general sort. Moreover, such principles can relate to a superordinate principle. The statement, "If a *number* series, then the sums may be obtained by sequential addition," in some sense refers to a principle of higher order than do the statements, "If an arithmetic series, then sum = $\frac{(A + L)_N}{2}$" and "If geometric, then $\frac{A-AR^{n+1}}{1-R}$," etc.[16] It is presumably this sort of knowledge that Gagne (1964, 1965) has in mind when he refers to problem solving.[17]

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[15] Composite operations appear to correspond to the notion of chaining in the S-R mediation language but as suggested above the former correspond to connections between mediating responses and their stimulus properties rather than between stimuli and responses.

[16] It is worth noting that a single principle may refer to the same stimulus class as do a number of more restrictive principles taken collectively. The principles, "If large and *black*, then (the response varies with) shape" and "If large and *white*, then shape," may be considered special cases of the principle, "If large, then color and shape." The latter, more general, principle at once has fewer identifying cues and more response determining dimensions than the other two. In effect, it appears that critical response dimensions are traded off with critical principle identifying cues. The more general the principle, the more stimulus attributes vary with the responses; the more specific the principle, the more stimulus properties are required to identify the principle. The total number of critical properties remains constant.

This discussion provides another powerful argument for adopting the principle, rather than the association, as the basic unit of behavior in meaningful learning. Not only is the principle able to account for structure, without postulating complicated chains and hierarchies of unobservables, but the principle has the important advantage of mathematical parsimony. When stripped of non-essentials, the set of observable stimuli and responses, which constitute the denotation of a principle, corresponds exactly to the mathematical definition of a function. The association, on the other hand, is a special case.

The role of directions. - Directions serve an important role in almost all experimentation with humans. With meaningful learning, this role may be critical (e.g., Maier, 1930; Gagne, 1962, 1964). According to Gagne (1964, 305), directions may serve to: (1) identify the terminal performance required, (2) identify parts of the stimulus situation, (3) aid the recall of relevant subordinate performance capabilities, and (4) channel thinking. The characterization of a principle as an ordered four-tuple (I, D, O, R) reflects each of these functions. R refers to the desired class of responses; I and D refer to the critical stimulus cues; and information about O may serve to aid recall and to channel thinking. Giving "complete" directions, of course, would amount to stating the principle.

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[17] In contrasting Gagne's classification scheme with the present formulation it is well to keep in mind the difference in purpose. Gagne (1964, 1965) was concerned with classifying learning according to the necessary and sufficient conditions for its occurrence. Present concern has been to show how the SFL provides a valid base for formulating each of these learning types. Although the preceding discussion suggests that the necessary preconditions for learning can also be derived from the SFL formulation, further discussion is beyond the scope of this paper.

In other situations, however, it appears that directions serve a higher order role. They seem simply to define the context, to limit the number of principles which might be evoked in the given situation. This function is served, for example, when S is told to find sums,thereby making it possible for him to restrict his attention. Such directions would appear to serve a motivational function; it is by such direction that psychologists attempt to manipulate the motivational factors involved in learning. Detailed consideration of the underlying mechanisms is beyond the scope of this paper but they would appear to be compatible with the SFL.

EMPIRICAL RESEARCH

The research reported in this section was formulated and completed prior to the development of the SFL in its present, but still preliminary, form. In the beginning, there was just a vague feeling that a new approach to meaningful learning was needed. The later research was based on a preliminary formulation of principles solely in terms of their denotations. I have tried to indicate both the chronology of this development and some of the pitfalls which helped to shape my thinking.

Pilot Research on Response Consistency. - During the summer of 1962, Greeno and Scandura (1966) found, in a verbal concept learning situation, that essentially S either gives the correct response the first time he sees a transfer stimulus or the transfer item is learned as its control.[18]

I later reasoned that if transfer obtains on the first trial, if at all, then responses to additional transfer items, under certain conditions, should be contingent on the response given to the first transfer stimulus. In effect, a first transfer stimulus could serve as a test to determine what had been learned during stage one, thereby making it possible to predict what response S would give to a second transfer stimulus.

To test this assumption, I had a number of pre- and post-doctoral psychologists over-learn a short list of S-R pairs like that shown in Figure 1. Prior to learning the list, both the Ss and E agreed on the relevant values and dimensions--size (large-small), color (black-white),

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[18] The learning (list 1) and transfer and control (list 2) stimuli were Underwood and Richardson (1956) nouns, the responses were nonsense syllables, and the lists were learned by a self-paced anticipation method. The transfer stimuli belonged to the same concept category as one subset of the learning stimuli.

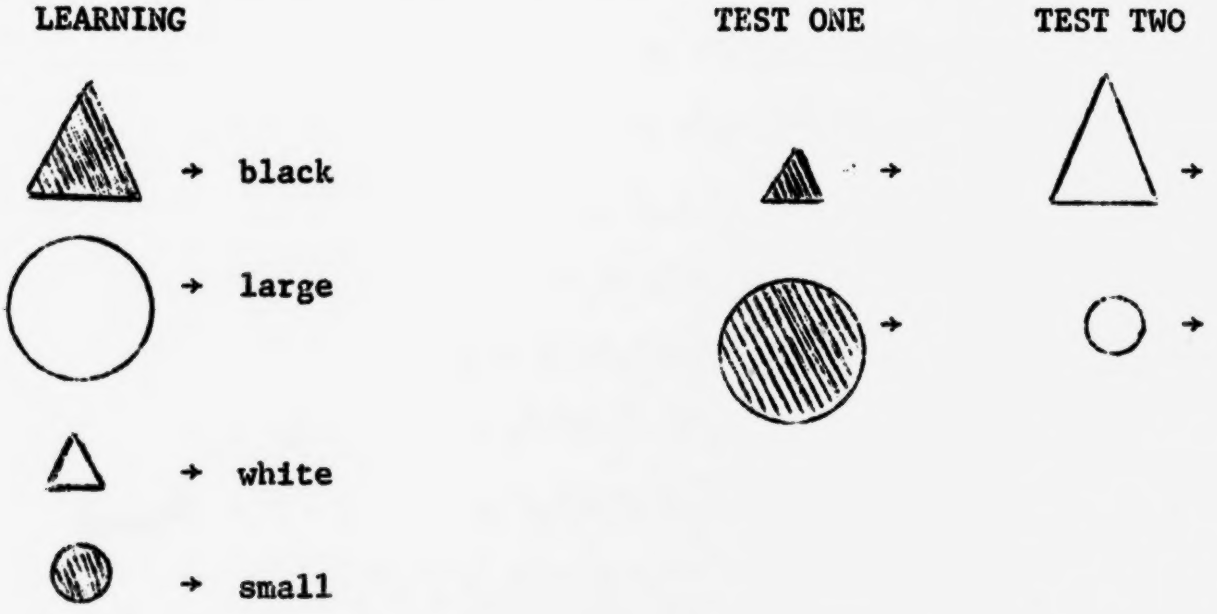

Fig. 1. Sample learning, assessment (test one), and prediction (test two) stimuli and responses.

and shape (circle, triangle). The Ss were told to learn the pairs in the most efficient manner possible as this might make it possible to respond appropriately to the transfer stimuli. After learning, the test one stimuli were presented and the Ss were told to respond on the basis of what they had just learned. Positive reinforcement was given no matter what the response. The test two stimuli were presented in the same manner.

The results were clear cut. Most of the Ss gave the responses, black and large, respectively, to the two test one stimuli (see Fig. 1.), and those who did, almost invariably responded with white and small to the test two stimuli. On what basis could this happen? It was surely not a simple case of stimulus generalization; the responses given did not depend solely on common stimulus properties. The first test one stimulus, for example, is as much like the fourth learning stimulus as the first (see Fig. 1.).

Perhaps the simplest interpretation of the obtained results, is that most of the Ss discovered the two underlying principles during list one learning and later applied them to the test stimuli. These principles might be stated, If (the stimulus is a) triangle, then (the response is the name of the) color" and "If circle, then size." The former principle is characterized by letting I = (triangle, color), D = R = (color), and O = identity mapping. The corresponding denotation would be [(S, R)|S is a colored triangle and R is the name of the S color].

The results obtained in this miniature pilot experiment (which I have repeated a number of times) provide support for the contention that principle learning is an all-or-none affair. The Ss either learned the principles (S-R relationships) or they didn't; there was no difference in test one and test two performance. Not all of the Ss, however, learned the two principles indicated above. Apparently, some of the Ss ignored the similarities between the four pairs and learned them in rote fashion--i.e., as four distinct principles, each involving an object and the corresponding word response. Under such circumstances, random test performance would be anticipated.

The results of another pilot study, conducted at the University of Michigan during the summer of 1963 and reported by Scandura (1966), were also revealing. In this case, Underwood & Richardson (1956), high dominance nouns were used, nouns which elicited an adjective associate with a frequency greater than 50%. Eleven college Ss overlearned a list consisting of four pairs of stimuli representing four adjective categories. Both stimuli in a given category were assigned a common response. The four test one and four test two stimuli were also high dominance nouns selected so as to represent each adjective category. The task was put in the context of a game and S had the option of responding to the test stimuli with, "I don't know," when none of the four learned responses seemed appropriate. Without this control, appropriate responding to a transfer stimulus would have occurred by chance in about one out of four cases. Again, positive reinforcement was given for all choices.

The results with these concept materials were equally revealing. In those cases where transfer potential was indicated, the responses to the second set of stimuli conformed to prediction in 47 of 52 cases. Furthermore, when asked, all but two of these S̲s correctly identified the common adjective as the basis for their test responses.

Nonetheless, these results cannot be interpreted as unambiguously as in the first pilot study. All of the important stimulus dimensions, could not be identified a̲ p̲r̲i̲o̲r̲i̲. In concept learning experiments, using the Underwood and Richardson (1956) materials, it may be desirable to assume that the set of determining properties, D, equals the common adjective.[19] [20]

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[19] With actual subject matters assessment sometimes presents additional problems. In the first place, it is not always easy to specify uniquely the basis for an overt response. There is usually more than one path to the goal. Consider, for example, a situation, which occurs in the University of Illinois Committee on School Mathematics Program (Jack Easely, personal communication), in which S̲ is asked to compute 35 · 449 + 35 · 551 as rapidly as possible. S̲ can laboriously multiply 35 times 449 and 35 times 551 and then add the products or he can recognize this is an instance where the distributive principle would allow him to compute 35(449 + 551) = 35 · 1000 = 35,000. Clearly, it is not the sum alone which determines what is learned (i.e., the "way" in which the problem is solved), but the t̲i̲m̲e̲ it takes to respond. If the correct answer is given in a short time, the distributive principle was probably used. Giving the answer in a relatively long time would likely indicate the usual computational rule. In either case, the principle used would be reflected in the denotative r̲e̲s̲p̲o̲n̲s̲e̲ u̲n̲i̲t̲ chosen. If S̲ gives an incorrect answer or if the problem is so easy that there would be little time differential no matter how S̲ does the computations, further complications are introduced.

In short, the careful selection of test stimuli and responses is essential in order to assess knowledge. Ideally, these elements should be chosen so as to eliminate all m̲o̲d̲i̲ o̲p̲e̲r̲a̲n̲d̲i̲ but the one in question.

Principle Learning. - The question of relationships between S-R pairs seems so basic, and so obvious, that one wonders why it has not been studied extensively. Because it provides a simple context in which to contrast mediation and set-function formulations, the problem is described in some detail.

Consider a paired-associate (PA) context in which the relationships between four pairs are varied while the other factors are held constant. In Figure 2, such a manipulation is accomplished by selecting the two principles indicated by, "If black, then shape" and "If white, then size."

Insert Figure 2 about here

In the experimental list, two pairs correspond to each of the two

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Although probably not attainable, this ideal can be approached in many cases. Another problem involved in work with actual subject matters is that of complexity. More than one principle may, and usually does, enter into a single test response. To determine the learning underlying the response, it is often necessary to assess each principle individually, as in diagnostic work with school children.

In many test situations, there are few available responses from which to choose (as in True-False and Multiple Choice tests). Under these conditions, there are additional problems of assessment since there is a high probability of giving any particular response (by guessing) irrespective of learning type. A similar problem obtains in assessing concept learning. There are at least three ways of minimizing this problem: (1) present more than one test stimulus, (2) include appropriate controls for comparison (e.g., Greeno & Scandura, 1965), and (3) provide an alternative for guessing as was done in the pilot study described above.

[20] The assessment methodology employed in this research may be used in conjunction with two types of variable: (1) those which affect the probability of principle learning and (2) those which affect response consistency. Giving directions and presenting cues, hints, or other attention-getting devices provide examples of the former type of variable. The consistency with which S responds according to a learned principle may be influenced by feedback, as well as instruction variables, operating between the first and second test responses.

A study dealing with the effects on principle learning of cueing various relevant stimulus properties is currently underway in our laboratory.

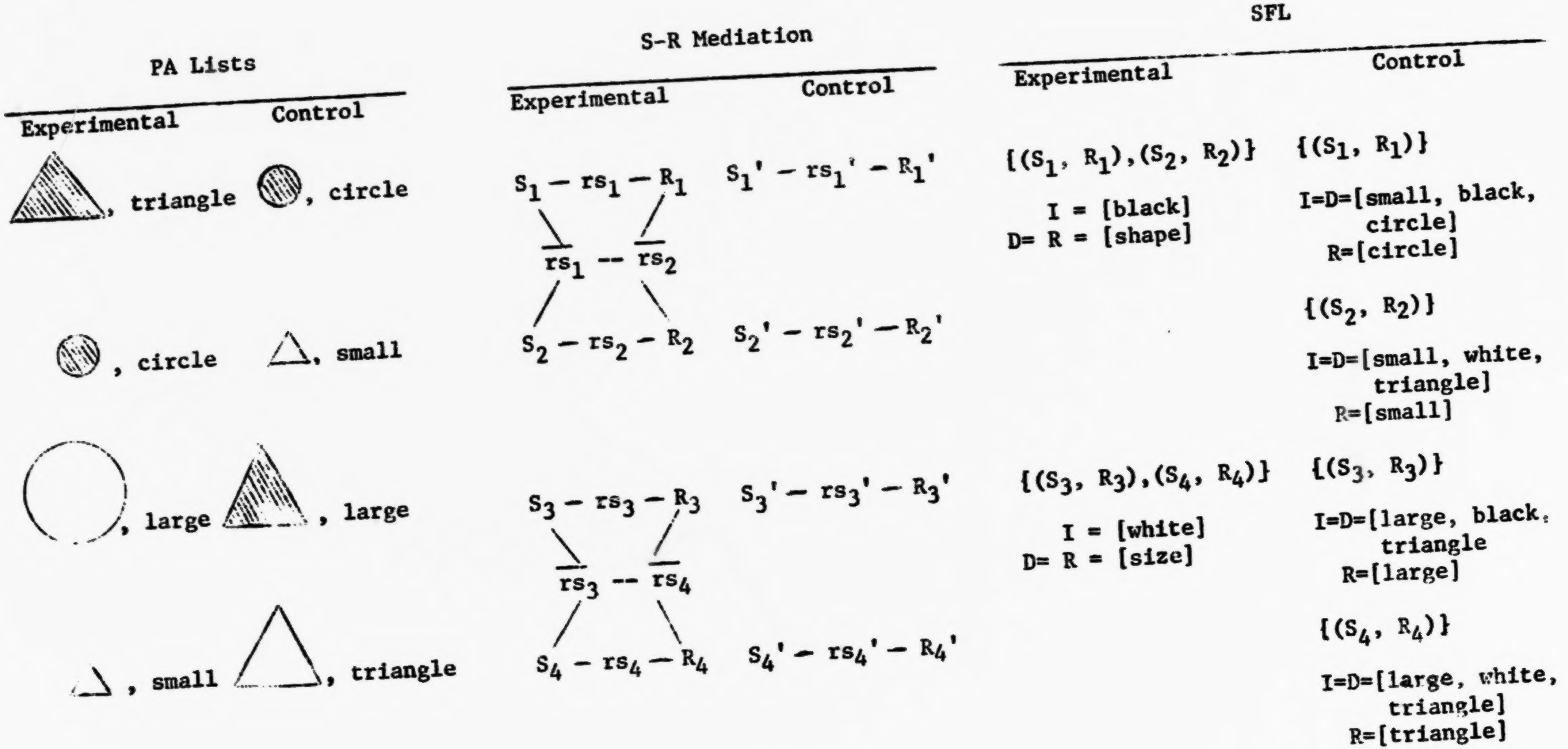
PA Lists
Experimental
Control
, triangle
, circle
, circle
, small
, large
, large
, small
, triangle
S-R Mediation
Experimental
Control
$S_1 - rs_1 - R_1$
$\overline{rs}_1 -- \overline{rs}_2$
$S_2 - rs_2 - R_2$
$S_1' - rs_1' - R_1'$
$S_2' - rs_2' - R_2'$
$S_3 - rs_3 - R_3$
$\overline{rs}_3 -- \overline{rs}_4$
$S_4 - rs_4 - R_4$
$S_3' - rs_3' - R_3'$
$S_4' - rs_4' - R_4'$
SFL
Experimental
Control
$\{(S_1, R_1), (S_2, R_2)\}$
I = [black]
D= R = [shape]
$\{(S_1, R_1)\}$
I=D=[small, black, circle]
R=[circle]
$\{(S_2, R_2)\}$
I=D=[small, white, triangle]
R=[small]
$\{(S_3, R_3), (S_4, R_4)\}$
I = [white]
D= R = [size]
$\{(S_3, R_3)\}$
I=D=[large, black, triangle
R=[large]
$\{(S_4, R_4)\}$
I=D=[large, white, triangle]
R=[triangle]

Fig. 2. Sample paired-associate lists, together with S-R mediation and SFL representations of these lists. In the experimental list the pairs are interrelated; in the control list, they are not. Two principles are involved in the experimental list: (1) If black, then shape, (2) If white, then size.

Broken lines indicate associations to be learned; solid lines indicate previous associations. The symbols, rs, refer to both the mediating response and the response produced stimulus. In the S-R mediation representation of the experimental lists, $\overline{rs}_1$ corresponds to "black," $\overline{rs}_2$ to "shape," $\overline{rs}_3$ to "white," $\overline{rs}_4$ to "size," rs_1 to "triangle," rs_2 to "circle," rs_3 to "large," and rs_4 to "small." In the control list representation, rs_1' corresponds to "circle," rs_2' to "small," rs_3' to "large," and rs_4' to "triangle."

In the SFL representation both the principles and denotations are characterized. O, in each case, is the mapping between D and R.

principles. The control stimulus properties (sizes, colors, and shapes) and responses (shape and size names) are identical with those in the experimental list. In addition, the responses, in both lists, are names of the corresponding stimulus properties. Any differences in the learnability of these lists would be hard to attribute to anything but the presence of relationships between pairs in the experimental list.[21]

Assuming S and E agree on the relevant stimulus dimensions, S's task in learning the experimental list can be viewed as that of discovering the principle identifying (I) and response determining (D) attributes since O is the identity map between D and R. On the other hand, S could learn the experimental list without noting any relationships between the pairs. Only one alternative is available in learning the control list; there is no principle involving more than one pair. To the extent that relationships between pairs are noted, the experimental list should be easier to learn.

The mediation description of the list contingencies in Figure 2 leaves much to be desired. The representation of principle learning is relatively complex and would have been even more so had we not let "rs" represent the typically made distinction between mediating responses and their assumed stimulus properties. No single chain, for example, can adequately represent principle learning in which more than one pair is involved.[22] The 1-1 pairing between the S_i and R_i ($i = 1, \ldots, 4$) does not follow from an analysis of the S-R links in the longer three stage chain. The chain does not make clear, for example, why R_1 is the response to S_1 rather than R_2. The more direct two-link chains involving the rs_i ($i = 1, \ldots, 4$) serve this purpose.

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[21] It may appear that an appropriate control list could be constructed by pairing the experimental stimuli and responses in random fashion. Alas, this turns out not to be a critical control. Any differences between the groups could then be attributed to pre-experimental associations (in the experimental group) between stimulus properties and the corresponding responses (shape names) rather than to relationships between pairs.

[22] The S-R representation proposed is original with the author as far as can be ascertained.

In view of this complexity, perhaps the most crucial limitation may prove to be the inability of S-R formulations to lead one to ask practically important questions concerning meaningful learning. The S-R representations that would seem to be called for bear more than a passing resemblance to pre-Copernican epicycles and related attempts to salvage geocentric theory.

With the assistance of Judith Anderson, I (Scandura, 1967) conducted a pilot study that is relevant. Its purpose was to determine relationships between the number of S-R pairs per principle, in a PA list, and learning rate and transfer.

The materials to be learned consisted of 12 pair lists. Each stimulus had a property relating to shape, border, shading, outline, and color. Four colors and eight values of each of the other four attributes were used. The responses were descriptive labels attached to the non-color stimulus properties (e.g., circle). Of the 12 pairs in each list, six were instances of one principle (P6), three were instances of another (P3), two were instances of a third (P2), and one was an instance of a fourth (P1). The principles were constructed so that the same principle applied to all stimuli having a particular color. The response determining cue was either a shape, a border, a shading or an outline. The four colors and the determining attribute dimensions (e.g., shape) were randomly paired to form four principles (e.g., If black, then shape), which appeared equally often under each condition. The PA list was learned by the anticipation method to a criterion of three consecutive errorless trials.

To determine whether the principles were acquired sometime during the list learning, each S was shown two transfer lists of four new stimuli each, eight in all. Each transfer list included one stimulus, associated with each of the four learning principles. Responding according to one of the principles was presumed to indicate that that principle had been learned.

Prior to learning the original list, each of the 20 college Ss was pre-trained so that he was familiar with the stimulus dimensions and could name each stimulus property. These responses were typed on a card and were always available to S. In addition, S was told that a pattern was involved which might facilitate his learning and guide his responses to the transfer stimuli.

The dependent variables were the average number of errors per instance (i.e., an S-R pair associated with a principle) for each S (on each of the four principles) and the number of appropriate responses to the transfer stimuli.

Except for a very small reversal between treatments P3 and P2, learning rate (i.e., the average number of errors per instance) decreased with the number of instances per principle: 5.0, 3.4, 3.5, and 2.7, respectively ($F = 8.76$, $df = 3/76$, $p < .001$). The difference between P1 and P2 was significant ($F = 11.50$, $df = 1/76$, $p < .01$) but none of the other adjacent means differed significantly. Under the experimental conditions, the rate of learning an S-R pair increased with the addition of a second S-R instance but increasing the number of instances still further apparently had little effect.

The number of appropriate responses to the transfer stimuli was also affected by the number of instances per principle. There were 27, 8, 15, and 9 appropriate responses (as indicated by the experimental principles) given to the P6, P3, P2, and P1 transfer stimuli, respectively. Although the trend was not entirely regular, a sign test indicated that the degree of principle learning was higher in treatment P6 than in the average of treatments P3, P2, and P1 ($z = 2.6$, $p < .005$).[23]

Another analysis demonstrated that P6 transfer was related to learning rate. Of those 9 Ss who responded appropriately to both P6 transfer stimuli, 7 had below median (2.61) error scores, indicating more rapid learning; of those 11 Ss who responded appropriately to at most one test stimulus, 8 had above median error scores, indicating slower learning. An exact probability test (Finney, 1948) on the resulting 2 x 2 contingency table indicated a significant relationship between P6 transfer and learning rate ($p < .035$). The small number of Ss who gave two appropriate responses with respect to the other principles precluded the possibility of obtaining significant relationships. Only 3, 5, and 2 Ss gave both desired responses to the P3, P2, and P1 test stimuli, respectively.

The list learning and transfer results were not entirely consistent. The inclusion of more than two instances did not affect learning rate, but it may have affected transfer. These results could reflect real differences or be simply artifacts of the situation. In either case, the overall pattern of results was sufficiently clear to make any interpretation in terms of stimulus or response generalization extremely difficult, if not impossible. Some resort to S-R generalization (Hull, 1943;

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[23] It might be argued that the difference in the number of appropriate responses was due to there being more responses per category in treatment P6. When in doubt, the Ss may have tended to give a response from the most frequently experienced category. A comparison, however, of the average number of P6 responses given to the P3, P2 and P1 transfer stimuli (16) was not significantly higher than the ten P3, P2, and P1 responses given to the P6 stimuli ($p > .10$).

Berlyne, 1965) would appear necessary.

Principle Generality, Learning, and Response Consistency. - Subject matter specialists, particularly those in mathematics, are inclined to emphasize the importance of teaching general, rather than specific, principles. Experimental data, on the other hand, indicate that the more closely learning and test stimuli approximate one another the better the test performance.

Armed with the denotative characterization of a principle, as a set of ordered pairs, Woodward, Lee, and I (Scandura et al, 1966) set out to reconcile opinion with apparently discrediting fact. In particular, we were concerned with the effects of principle generality on learnability and transfer. In the same study, we also explored the response consistency hypothesis with more complex materials. Two experiments were conducted, the independent variable in both cases being the scope of a principle statement. Scope was defined in terms of the corresponding denotation, one statement being more general than another if the denotation of the former included the latter.[24]

Our original hypotheses were that: (1) the scope of a principle would be fully reflected in performance, there would be little success with extra-scope problems and no differences in performance on within-scope problems, (2) the learnability of a statement, as determined by within-scope performance, would depend on scope, and (3) the combining rule taught would be used on all problems under conditions of non-reinforcement.

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[24] Notice that defining a denotation as a set, makes it possible to consider a variety of other relationships between different denotations. In particular, two denotations may be discrete (have no instances in common), overlap, or identical in addition to being ordered (one being more general than another).

In the first experiment, each group of 17 college Ss was presented with one of three ordered principles dealing with a number game called NIM (Banks, 1964, 55-58). In the game, two players alternately select numbers from a specified set of consecutive integers, beginning with one, and keep a running sum. The winner is the one who picks the last number in a series with a predetermined sum. If this sum is 31 and the set consists of the integers 1-6 the players alternatively select any number from 1-6 until the cumulative sum is either 31 or above (in which case no one wins). There is a compound rule which allows the player who goes first to win any such game, "Divide one more than the largest number in the set into the desired sum--make the remainder the first choice--on subsequent tries, consistently select that number which when added to the opponent's preceding choice sums to one more than the largest number in the set."

Each such game could be characterized by an ordered pair of integers. The application of each principle was illustrated with a common (6, 31) game. The least general principle (S), adequate for winning only (6, 31) games, was stated, "...make 3 your first selection. Then...make selections so that the sums corresponding to your selections differ by 7." Principle (SG) was adequate for solving (6, j) games j = 1, 2, ..., n and was stated, "the first selection is determined by dividing the desired sum by 7. ...Then...make selections so that the sums corresponding to your selections differ by 7." The most general principle (G) was adequate for solving (i, j) games i = 1, 2, ..., m; j = 1, 2, ..., n and was stated, "the first selection is determined by dividing the desired sum by one more than the largest integer in the set from which the selections must come. ...Then...make selections so that the sums corresponding to your selections differ by one greater than the largest integer in the set from which the selections must come."

All Ss, including two control groups, were tested on three problems. The first was within the scope of each principle, the second within the scope of all but principle S, and the third only within the scope of principle G.

The results were straightforward. Of those 13 Ss in group S who solved problem one, none solved problem two, and only one solved problem three. The corresponding numbers for groups SG and G were, respectively, 5, 4, 0 and 5, 5, 4. Within the scope of each principle there were only chance differences in performance on the problems. On the other hand, only one S solved an extra-scope problem.

The relative interpretability of the three rule statements was determined by comparing group performance on problem one which was within the scope of each. Rule S proved to be easier to learn, under the self-paced conditions, than did the rules SG and G ($p < .007$ in both cases). There was, however, essentially no difference in the interpretability of rules SG and G.

The third facet of this research was concerned with the consistency with which presented principles are applied. We wanted to determine whether the S and SG Ss would use the rule taught even when it was inappropriate (on the second and third problems). To make this possible, no information was given as to when the principles were and were not appropriate. In effect, the properties belonging to the set, I, were not specified--the "if" statement was identical for all three rules.

Of the 17 S Ss, 13, 9, and 8 used the rule taught on problems one, two, and three, respectively. The corresponding numbers in groups SG and G were 7, 7, and 5 and 6, 6, and 6. Although there was a slight tendency to not use the rules taught on problems two and/or three, there were no significant differences in frequency of use.

These results certainly provided strong support for our original hypotheses: (1) performance on within-scope problems did not differ appreciably, even though the common illustration was more similar to problem one than the others, and successful problem solving was limited almost exclusively to within-scope problems, (2) rule S proved easier to interpret than rules SG and G, and (3) the rules taught tended to be used consistently on all problems whether they were appropriate or not. The first mentioned result is particularly interesting since it tends to cast doubt on the assumption that there is a generalization gradient associated with S-R generalization and therefore provides indirect support for a rule interpretation (e.g., Berlyne, 1965, 171-174).[25]

About the only major unanticipated result in experiment one was that rule G proved as easy to interpret as rule SG. In view of the rather low proportion of successes in these groups, we were originally tempted to attribute the lack of such an effect to scale insensitivity.

To determine the generality of these findings, a second experiment, dealing with arithmetic series, was conducted with junior high school Ss. In this experiment, both scope (S, SG, G) and example (present, absent) were varied independently. One of the basic ideas, adequate for summing arithmetic series, repeated addition, was already familiar to most of the Ss and it was felt that an illustration of a rule might provide a basis for generalization, via discovery, to extra-scope problems. Another difference between this experiment and the first was that rule S, 50 x 50 (=2500), was effectively an answer given treatment and applied to only one series. It was used both as the common example and as problem one. In experiment one, rule S applied to a class of games.

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[25] Even if a generalization gradient is eventually demonstrated, S-R interpretations will have to consider the possibility that the result is simply an artifact resulting from the use of continuous dimensions and individual differences in "just noticeable differences" (e.g. see Lykken, Rose, Luther, & Maley, 1966). To the extent that the variables involved in meaningful learning are discrete, a rule interpretation may prove more useful.

Although the pattern of results shown on Table 1 paralleled those of experiment one in most respects, there were several important differences. First, the presence of the example (problem one) along with rule S resulted in significantly better performance on problem two than when rule S was shown alone. This was the only case in both experiments where non-negligible success was noted on an extra-scope problem. Using the analytical tools described above, it is conceivable that this effect was due to an important conceptual difference between the rule S statement and the others. "50 x 50" is clearly an instance of the more general combining rule, "n x n = n^2." This was not the case for any of the other rules. Presumably, the statement of rule S, together with the common illustrative series, 1 + 3 + 5 + ... + 97 + 99, provided the successful

TABLE 1

Summary, number of correct sums (uses of rule taught) on problems one, two, and three

	Rule				Rule and Example			
	N	one	two	three	N	one	two	three
Group S	20	8(8)	1(1)	2(1)	21	20(20)	9(0)	3(0)
Group SG	20	5(5)	4(5)	0(5)	15	11(12)	8(9)	1(6)
Group G	19	3(5)	5(7)	2(7)	19	18(18)	14(16)	3(15)

S Ss with enough cues to generalize. In particular, they may have discovered that this series had 50 terms. Hindsight suggests that this difficulty could have been overcome by simply stating the sum, 2500, of the illustrative series rather than "50 x 50."

Second, only three of the 19 G-with-example Ss solved problem three whereas 18 solved problem one and 14 solved problem two. The decrement between problems two and three was significant ($p < .004$). The reason for this difference was not immediately apparent. An intensive post hoc analysis of the test papers, however, suggested that the result may have been due to a difference in the derivation rule for determining the number of terms, N, for use in the combining rule, $\left(\frac{A + L}{2}\right)_N$. N could be determined from problem series one and two by taking the average of the first and last terms. This procedure led to an incorrect value (25, rather than 24) for N in the third series, $2 + 4 + 6 + \ldots + 46 + 48$. In short, the difficulty was not in the rule itself but in finding the correct value of N. Such difficulties may be circumvented in future experimentation by controlling for unwanted differences involving identifying properties, D (and/or combining rules, O).

Third, the results of experiment two cast further doubt on the hypothesis that the interpretability of a principle statement depends solely on the generality of its denotation--although it may tend to covary with generality. As indicated above, the more general the principle, the more abstract the properties and rule used to characterize it. Abstractness, of course, refers to the hierarchical level of the set properties--one property being more abstract than another if it refers to a collection which includes, as elements or derivatives thereof, sets to which the less abstract properties apply.

The interpretability of principle statements ("then" clauses in the present study) of varying generality may depend on both abstractness and the way the properties in I, D and R and the operation O are represented. Thus, making operational use of the relatively abstract arithmetic series property, "the difference between adjacent terms in some common value," necessarily presumes that, "the difference between adjacent terms is *two*," "...*three*," "etc.," can all be correctly interpreted. The converse, however, does not necessarily follow. Interpretability may also depend on the symbols actually used to denote the critical properties (in the sets I, D and/or R and/or O). Whereas "N" might suffice for one *S*, another might require, "the number of terms in the series." Both have the same referent, but the former symbolizes the latter expression more succinctly. Similarly, one *S* may be able to "compute $\frac{(A + L)}{2}N$" whereas another could not, requiring instead a statement like, "add A to L, then, divide the resulting sum by 2, and finally, multiply the quotient so determined by N." The latter rule statement simply makes clear the sequence of steps and binary operations implied by the algebraic statement.[26]

These observations lead to the following tentative definition of *description level*. *Symbolic representation A is of a higher level than B with respect to some reference symbolism (usually the native language) if the translation of A into the reference symbolism requires all of*

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[26] Why one way of symbolizing a statement is more interpretable than another, rather than vice versa, is a difficult question to answer, but it probably relates to the order in which symbolizations are learned (i.e., the native language first). Ordinarily, shorter statements are substituted for longer ones as their use becomes more frequent. Perhaps this is a natural process resulting from man's tendency to recode information into a manageable number of chunks (e.g., Miller, 1956). At any rate, it is to be expected that the shorter and simpler the representation, the more easily will it be learned and remembered.

those reference symbols needed to translate B plus some additional symbols of its own. Statements generally can be only partially ordered as to description level.[27]

In addition, since both abstractness and description level may vary concurrently, not all statements, referring to the same principle, are necessarily comparable as to *interpretability*.

Fourth, only one of the *Ss* who was shown the rule, 50 x 50, applied it to problems two and three. This result can probably be attributed to an interfering effect due to prior familiarity with addition problems. The *Ss* may simply have mistrusted rule S. How could a rule, like 50 x 50, having only one answer, be the sum of all three problem series. Most junior high school *Ss* would find it unreasonable that the series 1 + 3 + ... + 99 (problem one) and 1 + 3 + ... + 79 (problem two) have the same sum (50 x 50). Some such reluctance may also have obtained on problem one with group S-without-example. Nonetheless, we were surprised that only 8 of those 20 Ss, not presented with the illustrative series, gave the correct sum (2500 or 50 x 50) for test-series one.

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[27] Although they were not directly involved in the present study, there are at least three other ways in which principle statements may differ as to interpretability. First, the form of the statement may differ. Thus, instead of "If A, then B," a principle might read "B whenever A." Such differences could be important, especially at early developmental levels.

Second, the determining properties may be partially ordered as to discernibility. *Property A is said to be more discernible than property B if the determination of B requires the determination of A along with the determination of other properties or the use of one or more additional combining operations*. In this sense, the first term, A, in an arithmetic series is more discernible than the number of terms, N. Determining N requires the discrimination of A and all of the other series terms plus counting.

Third, combining operations may also differ as to interpretability even when their domains and ranges are identical. *Rule A is said to be more interpretable than rule B if rule A is equivalent to rule B but the simplification of rule B (to produce the desired response) necessarily involves A*. The combining rule $\frac{b^2-a^2}{2}$, for example, is more interpretable than the rule $\int_a^b t\,dt$, where the parameters a and b and the determined value are the same in both cases. The simplification of $\int_a^b t\,dt$ $(= \frac{b^2-a^2}{2})$ involves the algebraic rule $\frac{b^2-a^2}{2}$. How broadly applicable this definition is is an open question, but it would appear to hold promise as a first approximation.

The most important feature of these exploratory experiments was not the results, but the post hoc analyses they made possible. These analyses have led to what I believe is an improved formulation of the SFL, in general, as well as of the problem of statement interpretability, in particular. Most important, the present findings attest to the importance of strictly subject matter considerations in behavioral research. A growing body of research on meaningful learning (e.g., Gagne, 1961, 1962 Scandura, 1966, 1966, 1966; Roughead & Scandura, 1967) suggests that structural considerations are often more crucial than strictly behavioral variables such as amount of practice, massing, etc.

SFL ANALYSES

In this section, four problem areas have been selected for SFL analysis: (1) exposition and discovery learning. (2) reversal and nonreversal shifts, (3) conservation tasks, and (4) syntactic and semantic learning. Each of these problems has posed considerable difficulty for S-R analysts. To my knowledge, only the reversal-nonreversal shift problem has been successfully formulated in S-R terms (e.g., Kendler & Kendler, 1962). Whether or not an S-R formulation can ultimately be derived, however, is not in question. My point is simply that the SFL is up to the task and that I doubt that S-R formulations, even if devised, will be equally as helpful in generating fruitful research hypotheses.

Exposition and Discovery. - Previous studies involving expository and discovery modes of instruction have not been entirely consistent, even when the experiments, themselves, have apparently been well controlled (e.g., Craig, 1956; Gagne & Brown, 1961; Haslerud & Meyers, 1958; Kersh, 1958; Wittrock, 1963). These discrepancies have been due, in no small part, to the inadequate specification of underlying constructs. By helping to strip expository and discovery contexts of unessentials, SFL analyses may suggest important similarities and differences between treatments. For the sake of brevity, I shall not attempt comprehensive coverage of all possible expository and discovery treatments, but simply to give enough of the flavor of what is involved so that other analyses may be inferred.

Learning by exposition may take several forms, most of which can be reduced to interpreting statements of given principles. The scope of such statements (i.e., the inclusiveness of the corresponding denotations) may differ greatly. In some expository treatments, the responses corresponding to distinct stimuli are stated directly. In effect, the principles involved all have one pair denotations. Such a treatment is typically referred to as "answer-given" (e.g., Wittrock, 1963). In at least one study (Gagne & Brown, 1961), the answers given were formulas for finding sums of series rather than responses of the sort that have more typically been used, namely sums. Such apparent differences have led to confusion and have made direct comparisons difficult, if not impossible. Typical characteristics of this type of exposition are that the sets I and D, in the principle statements, are equal and the combining operation O, effectively maps a single experimental stimulus into a single response. In other expository treatments (e.g., Craig, 1956; Haslerud & Meyers, 1958; Scandura *et al*, 1965; Wittrock, 1963) more general principles (often called rules since the "if" clause is usually left to be inferred) have been used.

The criterion measures, used in conjunction with such treatments, have included examples used during learning to illustrate the principles (learning test), performance on new instances of the principles taught (transfer), and performance on new instances of new principles (nonspecific transfer).[28] In each case, test performance can be predicted on the basis of what has been learned. Performance on the illustrative items, of course, simply reflects the degree to which *they* are learned. Similarly, performance on new, but within scope, items reflects the degree to which the corresponding principle has been learned (i.e., interpreted correctly). Satisfactory performance on new instances of new principles is to be expected only if either the principles presented provide a basis for generalization (as the rule "50 x 50" provided a basis for generalization to "n x n") or other appropriate, but usually unidentified, modi operandi have been previously acquired.

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[28] Although frequently measured, retention is not considered in this analysis.

Discovery learning typically involves presenting stimuli or instances of principles and calling attention to critical stimulus cues, combining rules, and/or relationships between the stimuli and responses. In most experimental studies involving discovery learning, the learner is shown one stimulus at a time and is given varying amounts and kinds of directions as to how he might determine the correct response. The learner may be given hints which direct his attention, provide analogies, etc., as to what the critical (stimulus) properties, in sets I and D, are and/or how those in D may be combined to determine R. The more information given, presumably, the more likely is discovery to occur. Typically, a good deal of cue selection and combination is left to the learner.[29]

In comparing learning by exposition and discovery, some investigators (e.g., Craig, 1956; Haslerud & Meyers, 1958; Wittrock, 1963) have been concerned solely with the manipulable aspects of the stimulus situation. The occasion for reception or discovery learning is provided first and, then, followed by a test(s) for transfer. The results of such testing, however, are typically biased by uncontrolled differences in original learning. Since the relative ease of learning by exposition and discovery undoubtedly depends on the material in question, this approach can hardly be expected to achieve definitive results insofar as transfer is concerned.

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[29] Another form of discovery is often used in the classroom. In this case, S is presented, in turn, with several instances (S-R pairs) of a principle, with or without hints, and, then, is asked to give the correct response to the stimulus member of a new instance (Scandura, 1967). In effect, the learner is required to abstract a common combining rule. As indicated earlier, this process involves determining the properties in I and D and, in turn, discovering the common derivation rule for going from D to R.

Other investigators (e.g., Gagne & Brown, 1961) have been careful to equate original learning before testing for transfer, but they are open to criticism for not making explicit the relationships between their treatment and criterion measures. In particular, the relative advantages of reception and discovery learning depend on what is presented, on what criterion measure the groups are equated, and on what measure they are compared.[30]

Since the study was well designed, apparently definitive, and controlled for original learning, an analysis of the Gagne and Brown (1961) study is particularly instructive. In that study, the Ss were presented with number series, such as $1 + 2 + 4 + 8 + 16 + 32 + \ldots$, and were either given or required to discover algebraic formulas, involving the number of terms, n, for summing any series beginning with the specified terms and continuing with the same pattern. After completing a preliminary program, designed to acquaint the Ss with the concept of a number series and a number of terms relating to such series, three treatments were given. One group (RE) was presented, in turn, with the correct formulas, another group (GD) was given indirect guidance as to how to find such formulas, and a third group (D) was instructed to find formulas and was given hints as to what the formulas were like.

Since the error rate on all programs was reasonably low, it may be assumed that the three groups learned the desired rules to about the same degree. That is, each S who completed one of the programs was probably able to write the correct formula associated with each series presented. The required learning was simply a set of discrete one instance principles.

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[30] Learning ability is often cited, by pedagogical enthusiasts, as a major advantage of discovery methods of instruction.

The combining operations, O, learned by the R & E group, were probably simple associations between the properties in D and those in R, leaving little opportunity for abstracting a general rule for determining formulas. The GD group, and to a somewhat lesser extent the D group, had an opportunity to discover a stimulus processing rule; a rule which may have made it possible for them to determine the formulas corresponding to the test, as well as the training, series.

One of my doctoral students, William Roughead, and I carefully analyzed the training and test series as well as the discovery treatments themselves and found that this was the case. The technique by which the discovery Ss were led to discover the training formulas was also applicable to the test series. There was, in effect, a common principle relating to both the training and test series, but only the discovery groups had an opportunity to discover it.

As might be expected on the basis of this analysis, the results obtained by Gagne & Brown (1961) were clear cut. The test performance of the discovery groups was reliably superior to that of the exposition group. Gagne and Brown (1961) originally indicated that it might be possible to guess what information would be helpful to the discoverer, but they could not specify exactly why this information enhanced discovery. The present discussion suggests that such answers can be obtained from SFL analyses. The relationships become explicit in terms of the principles, statements, and denotations involved. More important, there appears to be no reason to suspect that SFL analyses will prove any less useful in formulating future research in this area.[31]

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[31] William Roughead and I have recently completed a study in which "what is learned" in discovery learning was taught by exposition and in which the order of giving answers and providing an opportunity for discovery was varied. The preliminary results appear very interesting and they will be reported in the near future.

Reversal and Nonreversal Shifts. - Figure 3 characterizes reversal and nonreversal shifts. The objects shown vary on two dimensions, size (large, small) and color (black, white). These objects are shown in pairs and S is required to choose the correct alternative, indicated by + in Fig. 3.

The experimental paradigm involves learning to make two discriminations, the second after S can reliably make the first. The first discrimination is identical for both groups; the second depends on the treatment, reversal or nonreversal. On the first discrimination, in the example shown in Fig. 3, size is the relevant dimension, large being positive. Color is irrelevant. A reversal shift involves the same dimension, size, but the correct response becomes small. A nonreversal shift involves the color dimension, black being correct.

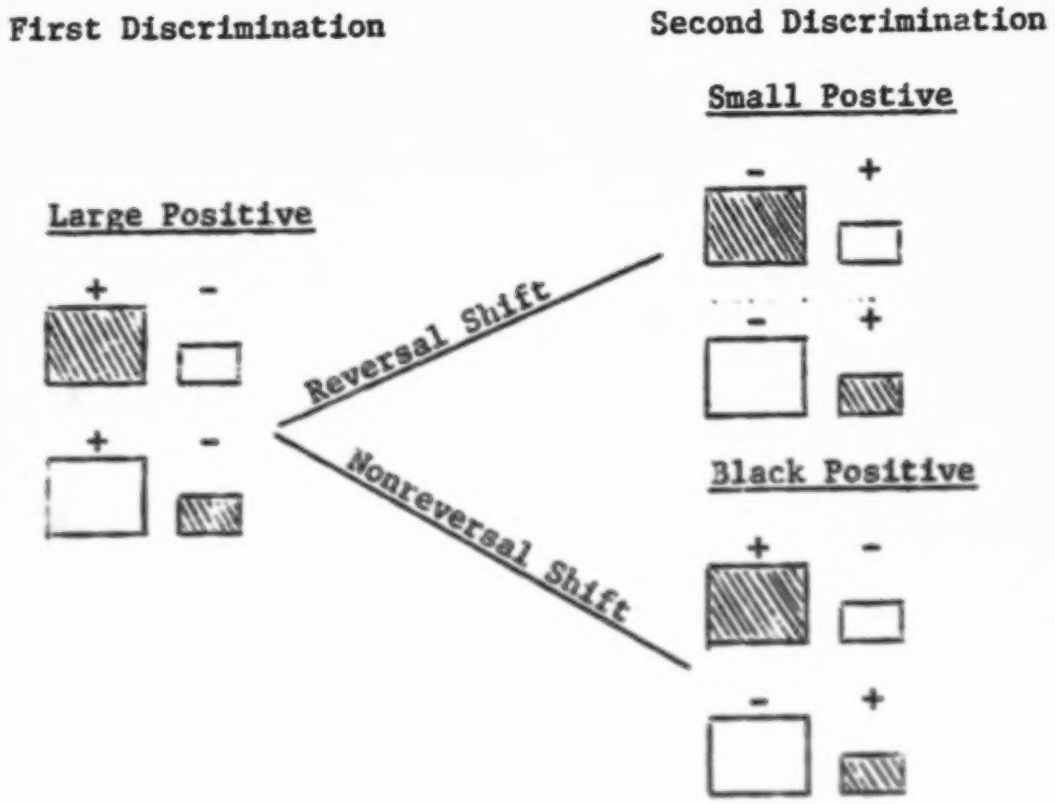
First Discrimination
Second Discrimination
Small Postive
Large Positive
+
-
+
-
Reversal Shift
Nonreversal Shift
-
+
-
+
Black Positive
+
-
-
+

Fig. 3. Example of a reversal and a nonreversal shift.

Preverbal children (Kendler, Kendler & Wells, 1960) and animals (Kelleher, 1956) find nonreversal shifts easier to make than reversal shifts whereas older, more verbal, Ss find reversal shifts easier (Buss, 1956; Harrow & Friedman, 1958, Isaacs & Duncan, 1962; Kendler & D'Amato, 1955). In a study with kindergarten Ss, Kendler and Kendler (1959) found that fast learners, like verbal Ss, were better able to make a reversal shift whereas slow learners, like preverbal Ss, were better able to make a nonreversal shift. It was suggested that the fast Ss approached the experimental task with verbal labels for the correct stimulus already strongly attached, the verbal label serving as a mediating link in a two-stage S-R paradigm. The learning of the slow and presumably preverbal Ss was assumed to involve a single-stage paradigm. Kendler and Kendler (1962) explained the relative ease of reversal and nonreversal shifts in terms of the number of S-R associations that need to be changed.

This interpretation, however, makes no provision for answering the question of whether the increasing ease of making reversal shifts with age is due to a gradual increase in verbal ability by all Ss or to some specific characteristic had by a larger proportion of faster learning children. Reformulating the reversal-nonreversal problem in the SFL provides a basis for answering this question.

Since this problem is different, in an important sense, from any previously encountered in this paper, some preliminary observations are in order. For one thing, the stimuli consist of pairs of objects. In addition, the critical stimulus (set) properties are relationships between objects.[32] Learning to make the first discrimination shown in

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[32] Stimuli having a common relationship need not have any "physical" property in common.

Fig. 3, for example, may be equivalent to learning a principle in which I = D = [one object larger than other], R = [larger object], and O is a rule which maps the relational property in D onto R.[33] Such a discrimination can also be accomplished by learning two or more less general principles. Notice that during the first discrimination task, S is never presented with two large or two small objects. Thus, S could learn to always choose the large black object when it appears and the large white object when it appears. The fact that both are larger than the object with which they are shown might, for example, go unnoticed with young children. One such less general principle might be stated, "If shown two objects, one of which is large and black, then the response is determined by choosing the large black object."[34]

Learning a single relational principle on the first discrimination should result in reversal shifts being easier than nonreversal shifts. In a reversal shift the critical properties would remain identical, only the operation, O, would need to be changed--pick the smaller, rather than the larger, object. A nonreversal shift would involve learning either a completely new general principle or two new more specific principles, in which both the critical cues (involving color) and operation need to be identified. On the other hand, if two less general principles are learned on the first discrimination, a reversal shift would involve learning two new subprinciples (or one more general principle) whereas a nonreversal shift would involve learning only one. The principle, indicated by the statement, "If two objects, one of which is large and black, then choose the large black object," is equally applicable to the original discrimination task and the nonreversal shift task in which black is positive (see Fig. 3). In short, the relative ease of shift may be dependent on "what is learned" on the first discrimination.

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[33] A principle of this sort is analogous to what Bruner, Goodnow, and Austin (1956) have called a relational concept. Although not treated here, there is reason to believe that transposition phenomena (cf. Hebert & Krantz, 1965) may also be reformulated in terms of relational concepts or principles.

[34] It should be emphasized that S need not be aware of verbal labels in order to learn a principle. The statements in the text serve primarily a definitional function.

Of course, this interpretation is analogous to that presented by S-R theorists (e.g., Goss, 1961); Kendler & Kendler, 1962). It is the implied assessment methodology which provides the means for determining whether "what is learned" is related to relative ease of shift. In order to determine Ss basis for making the first discrimination, it is necessary to employ dimensions, such as color and shape, which have more than two easily discriminated values. This procedure would make it possible to use two values of each dimension in presenting the first discrimination problem leaving the other values for assessment purposes. Suppose, for example, the four objects used in training are either black or shaded and a circle or a square. Suppose, further, that color is the critical dimension on the first discrimination. Then, the assessment procedure might involve a new discrimination problem in which the four objects used are describable in terms of the two colors (black, shaded) used during training, and a shape (e.g., triangle) not so used.

In order to help minimize "strategy shifts," between learning and assessment, positive reinforcement might be given at each choice point, no matter what the response. Under these and appropriate instructional conditions, choosing, as positive, the object having the same color as the positive object, used during training, would be indicative of learning a general principle (concept) on the training task were it not for the high probability (1/2) of choosing this object by chance alone. Assessment "certainty," of course, might be increased by using more test discrimination tasks. This would be made possible by increasing the number of values per dimension.[35]

Unfortunately, there is no relevant data. Because of the obvious implications for future research in this area, earlier verification of this analysis is urged.[36]

Conservation versus Non-Conservation. - Questions relating to the conservation (i.e., invariance) of such properties as amount and number (e.g., see Flavell, 1963), comprise another problem area in cognitive development that may be reformulated in SFL terms.

Consider a procedure that is frequently used to determine whether a child has learned to conserve amount. E shows the child two balls of clay, both of the same size as in display one of Fig. 4. Indeed, E may let S make them the same size. S is then asked whether each ball, in turn, contains more clay than the other or whether they contain the same amount. Invariably the normal child says they are the same. Next, E rolls one of the balls into the shape of a sausage as shown in display two of Fig. 4 and, then, asks the same question. If the child says that they are the same, and he does so consistently on this and other tasks, he is said to conserve amount. If not, he is a nonconserver.

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[35] It is possible to devise reversal-nonreversal type tasks involving more general principles and/or principles not involving relational stimulus properties. Under such conditions, assessment problems can be minimized.

[36] Since this manuscript was written, Tighe (1965) has published a paper in which she demonstrated that the relative ease of making reversal and nonreversal shifts with 5 and 6 yr. old children can be manipulated by prior training designed to emphasize the independence and dimensional nature of the properties of stimuli used in subsequent discrimination shifts. These results provide support for the current interpretation in that learning a general principle on the first discrimination requires that S recognize the independence of the two object dimensions. With the more restrictive principles only object, and not dimension, differences are important.

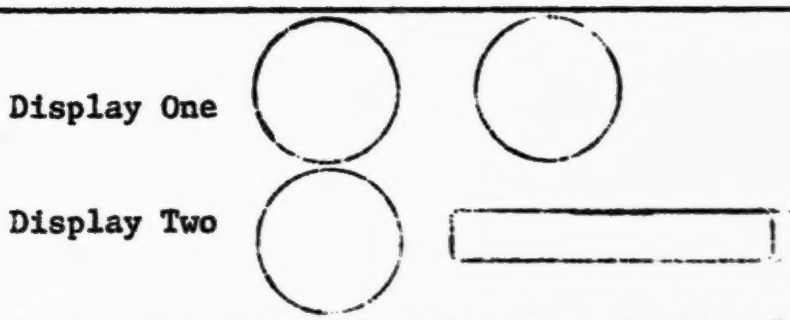

Fig. 4. Two displays designed to determine the conservation of amount. As with reversal and nonreversal shifts, the stimuli are object pairs and the critical properties are relational.

When formulated in the SFL, E is led to ask not merely whether S is or is not a conserver, but on what basis S is responding. The objects shown in the Fig. 4 displays are related in many ways--relative volume, weight, height, length, width, shape--besides relative amount. Any one of these (not necessarily independent) relational properties could provide a basis for responding. In display one, for example, the two objects contain the same amount and have the same length (as well as just about everything else except position). Thus, a correct response could signify the operation of either of the two principles, "If two clay objects, then the response depends on the relative amount," or "If...relative length." A child's reaction to display two may make it possible to determine which of these two principles is operating. Thus, if the child says that the sausage (Display Two) contains more clay, the operating principle probably involved length. If he says "the same," amount was likely the determining factor.[37]

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[37] It is worth noting that both of these two responses eliminate height from consideration as a basis for responding. Saying that the sausage contains more clay, however, does not eliminate shape.

The form of analysis, described above, is quite similar to that applied to reversal and nonreversal shifts. Both involve assessing what the child has previously learned. The essential difference is that, with reversal and nonreversal shifts, the relevant prior experiences (on the first discrimination) can be specified. In developmental situations of the sort described, conservation of amount is assumed to have been acquired or not acquired prior to the experiment. If S does not demonstrate conservation, it is presumably not a question of choice but of necessity. The principle of length may be operating, rather than that of amount, not because S prefers it, but because he has not acquired the concept of amount. A child, of course, may acquire both the concepts of amount and length in a generic, nonverbal, sense (e.g., Braine, 1959) before he knows what the words "amount" and "length" actually mean. All that can be said with confidence is that the phrase, "which contains more clay," may be interpreted by a child in any one of several alternative ways.

Syntactic and Semantic Learning. - All of the stimuli, as well as principle statements, considered so far are symbolic representations of an abstraction. A stimulus such as "1 + 3 + 5 + 7," for example, symbolizes an abstraction reflecting the structure of a variety of more concrete stimulus situations--e.g., four stacks of pennies, the first containing one penny, the second three, the third five, and the fourth seven; a figure representing the produce of four countries, ..., etc.

Suppose a young child has been taught a principle which makes it possible to write "16" when shown "1 + 3 + 5 + 7." What happens when he is presented with the four stacks of pennies and is asked how many there are? The answer to this question undoubtedly depends, to some extent, on the significance to *S* of the number symbols (i.e., numerals) in the symbolic stimulus. If the numerals refer to properties of collections of sets, each including a common number of elements or objects, positive transfer may be expected. If, on the other hand, the numerals and the arithmetic operations, such as addition, have been learned entirely without referents, say with flash cards, one could feel fairly certain that *S* would see no relationship.

Fortunately, this question can be formulated precisely by characterizing the principles involved in the SFL. Assume that the principle (there are other possibilities), corresponding to the symbolic stimulus, has been determined, by assessment procedures, to be (I = D = [number in first position, ..., ... fourth position], O ≡ repeated addition, R ≡ sums). The requisite for applying this principle, once learned, in a concrete situation is precisely that principle which makes it possible to go from the concrete situation to the corresponding numbers. A composite principle, consisting of this principle, along with that corresponding to the symbolic stimulus above, might be characterized (I = D = [number of *objects* in first position, ..., ...fourth position], O ≡ determine the number of *objects* in each pile (position) by counting the objects, perform repeated addition, and note that the sum denotes the total number of objects, R ≡ total number of *objects*). Learning the symbolic (former) principle, without being able to recognize its concrete referents, would be like having an egg shell, but no egg.

The relatively simple hierarchical SFL analysis proposed provides, I feel, but a prelude to the insights which may eventuate from similar analyses in other situations. Even partial clarification of the relative roles of symbolism and concrete referents in meaningful learning and performance could have important practical as well as theoretical implications and is long overdue. Although perhaps most directly applicable to mathematics, this sort of hierarchical analysis may also prove useful with other subject matters.

THEORETICAL COMMENT

This paper deals more with a precise scientific language (SFL) than with theory. Although certain problems have been partially clarified, any predictions resulting therefrom would be based not so much on new theoretical assumptions as on logical analyses of the situations involved.

Nonetheless, the identification of the principle as the basic behavioral unit, is bound to have important theoretical implications. A recurring theme of this paper is that the association is too restrictive a unit on which to build a theory of meaningful learning. Although close relationships have been shown between the association and the principle, each being derivable from the other, the representation of increasingly complex learning situations, in terms of associations, soon becomes prohibitively cumbersome. The principle, on the other hand, appears up to the task. Equally important, the principle not only has much in common with a number of proposed basic constructs, such as rules, schemas, and TOTE units, but the SFL representation of a principle provides cognitive theorists with a much needed ingredient--precision.

Although it is beyond the scope of this paper to present anything approaching a finely-spun theory, I will try to indicate two of the ways in which theory development, based on the SFL, might proceed. In the process, I hope to suggest some fundamental differences between statistical theories, designed to predict group behavior, and more deterministic theories, which make the prediction of individual behavior possible.

The first approach is well exemplified by, but certainly not limited to, stochastic learning models. In these theories, given an initial state each S is typically assumed on each trial to enter the next state with a certain probability. This process continues until the terminal or absorbing state is reached. To make predictions, based on such theories, the values of the transition probabilities, which presumably are based on underlying, physiological capacities, are estimated from data acquired in situations which are closely related to, and usually the same as, those in which the predictions are made. The predictions, themselves, deal exclusively with parameters of the resulting distributions of scores. In short, assumptions are made about individual learning processes and predictions are made about group behavior.

In order to see how such a theory might treat principle learning, consider a situation in which instances of a principle (i.e., related S-R pairs) are presented until S can reliably anticipate new responses to new, within scope, stimuli. To discover the underlying principle, S must (1) determine the relevant properties of the stimuli and responses and (2) discover the rule relating them. In addition, if noninstances are included in the test list, S would have to identify those stimulus properties which make it possible to discriminate between instances and noninstances.

These requirements imply a model extension of the sort suggested by Haygood and Bourne (1965). These authors postulated the need for a second process, rule selection, which is independent of attribute selection—the latter providing the basis for Restle's (1962) theory of concept learning (i.e., attribute identification). In the present, more general, situation, stimulus dimensions, rather than properties, would also need to be taken into account. The identification of the relevant response dimension would also be involved. Only under these conditions could principle learning be expected. Notice, in particular, that varying stimuli, although sufficient for concept learning, would not *necessarily* increase the likelihood of discovering a principle from its instances. The responses would also have to vary. These considerations imply a model with at least three, and possibly more, independent processes.

A major limitation of present-day stochastic models is that they are fundamentally incapable of predicting individual performance. All *S*s are assumed to bring with them the same learning parameters (i.e., transition probabilities); yet, it is common knowledge that the *S*s enter these situations with important experiential differences.[38] Such differences, of course, could be taken into account so that different groups of *S*s are assigned different transition probabilities. In the limiting case, individual parameters could be based on individual data. But, then, no rational theorist would resort to such nonsense—using an individual's data to predict *the* individual's data.

My reason for this discussion has not been to discredit stochastic models. Predicting group performance is equally as defensible, and important, as predicting individual performance. What I have intended is to provide a perspective for viewing present theoretical concerns.

Contrast the foregoing approach with a theory which says, in effect, that if the relevant things about *S*'s present state are known, it would be possible to predict what he would do in any given situation. Such a theory would necessarily be concerned with determining what the relevant state characteristics are and with procedures for determining whether they are within S's immediate repertoire. Rather than making assumptions about underlying physiological capacities (e.g., values of transition probabilities) to make predictions about learning, the proposed approach would involve inferring behavioral capacities in one situation to predict behavior in another. It would be a measurement based theory, unlike that exposed by present day neoassociationism, in which internal factors, determined from past behavior, would play the central role. Such a theory would need to specify how inputs interact with the state of an organism to produce responses. Scandura (1966) has argued that a theory of this sort would be, to a large extent, independent of traditional learning theory.

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[38] There are, of course, situations in which organismic differences may be minimal. The backgrounds of planaria and new-born babies undoubtedly differ less than that of normal adults. Similarly, humans are more likely to flutter their eyelids in the same way than they are to solve mathematical problems. Under such conditions, largely ignoring the effects of prior learning has caused relatively few difficulties.

The writing of many theorists (e.g., Müller, 1913; Hull, 1935; Berlyne, 1965) indicates that there are at least two aspects of state that may affect what response is given to a presented stimulus: (1) motivation and (2) prior knowledge. Although I am not prepared to defend the idea here, motivation might conceivably be viewed as a higher order principle of some sort which determines what unit of knowledge (i.e., principle) is to be applied. The principle, in turn, uniquely determines what the response is to be. Without some such mechanism, organisms would be incapable of responding in any reasonable fashion. Having been presented with an arithmetic series written on a piece of paper, S wouldn't know whether to count the number of terms, give the sum, or burn it (the paper).

Although the values of the state variables would necessarily have to be inferred from behavior, all empirical evidence would not have to be in the form of R-R relationships. State variables can be manipulated by prior training. Principle learning, for example, can presumably be manipulated, at least within certain populations, by presenting suitable statements directly or by discovery. Assessment procedures would play an essential role in such manipulations by insuring that the intended learning did, in fact, take place. Directions would appear to be the manipulable counterpart of internal motivation. In effect, a stage approach to research of this type could provide a basis for establishing causal relationships.[39]

Although clearly speculative, I would be remiss if I did not state explicitly what to date have been my working assumptions. First, regardless of how principles are acquired in the first place, behavior, depends on the selection and use of some principle. Second, motivational factors may be viewed as providing a context within which principles are selected. For example, the statement, "If number series, then sum," would serve to limit the class of potentially applicable principles by specifying both the relevant stimuli and the desired objective. Within this context, S's selection of a principle would depend on the stimulus properties previously denoted by I. Third, principles are assumed to continue in operation so long as the then present motivational state obtains. According to this postulate, responses to new stimuli remain under the control of a particular principle unless new input or feedback otherwise indicates that the rules have changed. A tentative fourth postulate is that motivational states may be subject to control by higher order motivational states.

Clearly, these speculative postulates do not, in themselves, constitute the basis for any theory. At a minimum, clearer specification is needed as to how new principles are brought into play and how existing knowledge is modified. Some speculations can also be made as to the mechanisms underlying such changes. Perhaps the simplest assumption is that a principle ceases to operate when it is no longer applicable--as when the stimulus input does not correspond to an instance of that principle. There are at least two ways in which such a situation may arise. The environmental situation may change as the result of the responses controlled by the operating principle or the change may be the result of extra-subject influences. Suppose, for example, that an S is in the process of nailing planking for the floor of a house. The operating principle is "If (the nail is) up, then hit (with hammer)." The very act of hitting causes the stimulus situation to change. This new stimulus, in turn, could serve to "call up" the next principle--e.g., "If down, move to the next nail," or "If down and plank secure, get a new plank." The introduction of extra scope stimuli or conflicting directions could also transfer response control to a new principle. It would be a brave (or stupid) individual, indeed, who did not cease hammering and run as the result of someone's shouting, "Watch out for the rattlesnake!"

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[39] This is not to imply that a stage approach has not been used before. Much of the research in the Hullian tradition, involving drive and habit strength, would fit this general paradigm. The research of many investigators (e.g., Hilgard, Irvine, & Whipple, 1953; Katona, 1940; Maltzman, Eisman, Brooks & Smith, 1956), who have been primarily concerned with meaningful learning, also has been conducted in stages. More recently, Gagne (1964) has argued pervasively for a stage approach to problem solving research.

Cognitive disequilibrium theories (e.g., see Piaget as reported in Flavell, 1963; Berlyne, 1965) suggest that a learned principle may be modified if the response dictated by one principle is made to conflict with that otherwise indicated. Suppose the learned and operating principle can be stated, "If furry animal with four legs, then kitty," and S is shown a furry animal with four legs which happens to be a squirrel. The dictated response would be "kitty." If, however, feedback indicates that the correct response is "squirrel," S is confronted with a dilemma. He can either retain his principle, (almost) in its original form, and remember the exception (as a new one-instance principle) or revise the original principle and assimilate the exception as an instance of another (perhaps new) general principle.

The theorist might attempt to reconcile the outcome of such conflict in one of at least two ways. He might, if he has a physiological bent, for example, postulate that the amount of cognitive strain (e.g., Bruner, Goodnow, & Austin, 1956; Miller, 1956), is crucial. If committed to a state theory, on the other hand, the theorist would presumably resort to higher order principles of some sort.

CONCLUDING REMARKS

The role of any scientific language is to capture the essence of an area—to provide a means for dealing with the relevant variables while making it possible to avoid irrelevancies. Stated in more operational terms, a language is useful to the extent that it leads the research worker to ask fundamental questions. If it is precise, so much the better.

In the earlier sections of this paper, it was argued that the SFL meets both of these criteria. The SFL was shown to be significantly broad to encompass a wide variety of behavioral phenomena, from rote to meaningful learning. The language is precise, can be formulated in mathematical terminology whenever necessary or desirable, and provides a basis for relating internal events to observables. In particular, the SFL, coupled with the response consistency hypothesis, makes it possible to consider the vexing problem of "what is learned." After assumptions are made regarding the underlying stimulus values and dimensions, learning can be defined in terms of observable test stimuli and responses. An assessment methodology of this sort may provide the necessary basis for the increasingly called for multi-stage approach to meaningful learning.

Perhaps most important, taking the principle as the basic behavioral unit makes it possible to deal effectively with actual subject matters and leads one to ask new and presumably important questions. Both the pilot research, that helped shape our thinking, and the analyses, described above, are indicative of the kinds of research suggested by the SFL. Generality, abstractness, prior learning, statement interpretability, and discovery--all matters which have long plagued thinking pedagogs as well as psychologists--have lent themselves to rather close scrutiny. In short, the SFL provides a rigorous basis for studying many, if not all, forms of meaningful reception and discovery learning and their interrelationships. Future research may help unlock many of the behavioral mysteries that have traditionally been attached to these ideas.

Nonetheless, the SFL is far from a finished product. Many things remain to be done. First, and foremost, the validity of the proposed analyses needs to be ascertained. The already completed SFL based research (e.g., Scandura et al, 1965) was based on a preliminary and largely inadequate version of the SFL. Theoretical implications need to be more carefully and completely drawn out and suitable empirical tests conducted.

Second, additional modifications, extensions, and implications of the SFL need to be considered. I am, for example, convinced of the desirability and feasibility of extending the SFL so as to formulate questions concerning programed learning, particularly as regards providing a rigorous basis for such technologies as task analysis (e.g., Gagne, 1962). There also is reason to believe that the cognitive processes, classified in

Bloom's (1956) taxonomy, may find precise behavioral counterparts in the SFL. Making these relationships explicit could have important implications for measurement.

Third, more rigorous attempts should be made to reformulate S-R mediation based phenomena and constructs, such as stimulus and stimulus-response generalization, discrimination, and classical conditioning, in SFL terms. Although this might not always prove to be possible, important insights into both languages may be so attained.

In this regard, I might add that I am under no delusions that the SFL can deal effectively with all that the older S-R language can. This need not be a crucial limitation, however, since their primary areas of concern are different (Scandura, 1966). It may be no more necessary for the SFL to deal effectively with all S-R phenomena any more than heliocentric theory needed to deal with epicycles. Many of the phenomena dealt with within a given theoretical framework disappear when looked at from a different point of view. Nonetheless, although apparently lacking at present, the S-R language may again prove sufficiently flexible to deal effectively with meaningful learning. This possibility must certainly be entertained.

Fourth, the question of memory has not been considered in more than a superficial way. If what can be learned in a given situation depends on certain physiological limitations, as has been suggested, then what happens to that information which is stored. I (Scandura, 1967; Scandura & Roughead, 1967) have suggested that what is stored isn't "forgotten." It may simply get mixed up with what is already there. Prior or subsequent activity and learning could result in the storage of competing information in the memory locations or in making it hard to relocate this location at recall. Whether such ideas can be formulated in SFL terms or not is a completely open, but apparently very complex, question.

Finally, more explicit consideration needs to be given to the relationships between the SFL, its related postulates, and existing cognitive theories. Since the notion of conceptual conflict is so fundamental to many cognitive theories (e.g., see Berlyne, 1965), it might provide a good place to start.

At this point, no one can argue with any authority as to the ultimate value of the SFL. The outcome is even less certain as regards theory construction. Nonetheless, the SFL appears sufficiently promising to warrant further investigation.

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THE BASIC UNIT IN MEANINGFUL LEARNING[1]--

ASSOCIATION OR PRINCIPLE?
(Set-Function Language)

Theoretical development in educational psychology has been extremely slow. One major reason has been the typically imprecise definition of independent and dependent variables in research on meaningful learning and teaching. Such research can bear only an ambiguous relationship to theory. Similarities and essential differences often go undetected. As McDonald has put it,[2] "Conceptual clarity means (a) specification, stated in terms as nearly operational as possible, of the behavior involved in a task or method; (b) some delineation of the range of phenomena included and excluded; and (c) precise description of the appropriate tests."

Stating research objectives and defining variables in unambiguous terms, however, is not sufficient. The teaching-learning process has all too frequently been studied in terms of administrative variables, such as class size, grade level, and teaching experience. The variables chosen must have broad explanatory potential, not be merely symtomatic of and inextricably related to the question at hand. Theory development depends on much more than mere fact finding.

To provide a substantive base for their research, educational psychologists have frequently resorted to the languages, paradigms, and theories of the mother science of psychology. Mediational elaborations and operant conditioning paradigms of the stimulus-response (S-R) language and more general, but less well-specified, cognitive theories have been popular.

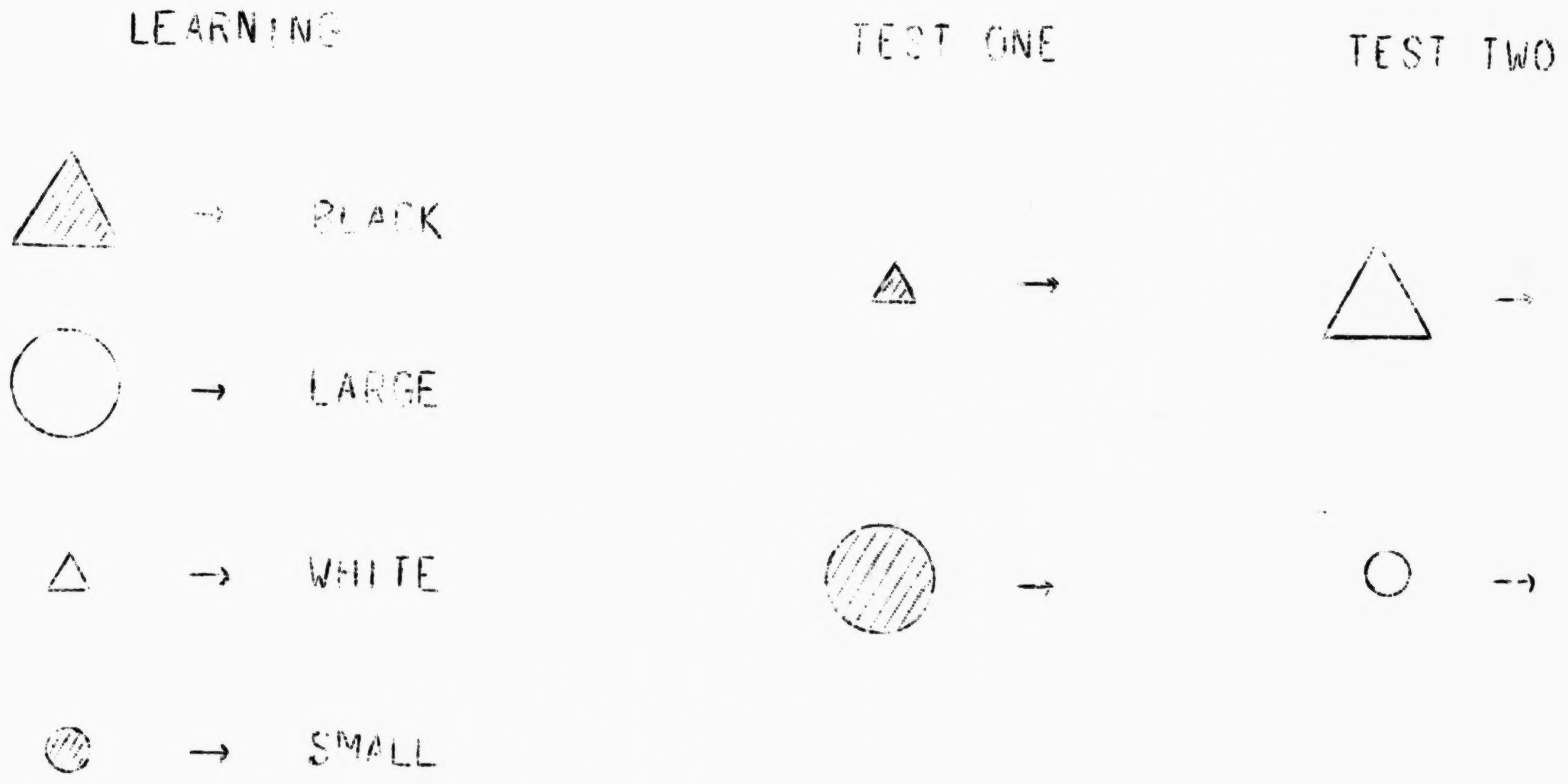
Each approach has important limitations. From one point of view, parsimony suggests that the properties of overt S-R associations should also be attributed to mediational links. Yet, practice has shown that mediational interpretations become increasingly cumbersome and less precise as situations become more complex. Similar difficulties have plagued researchers who have used operant techniques to study meaningful verbal learning. The results simply are no where near so clear in complex human learning as they are in the "Skinner Box." It is increasingly recognized, for example, that knowledge of results is not directly analogous to feeding a pigeon and that, in any case, other factors, such as subject matter structure, are probably of greater importance in promoting efficient learning. A general limitation of cognitive theories is their relative imprecision. Typically, "cognitions" are either not clearly specified in observable terms or are only partially defined.

In short, the choice to date has been between a precise, but seemingly inappropriate, S-R language and presumably more relevant cognitive formulations which leave much to be desired insofar as scientific cohesiveness and rigor are concerned.

The purpose of this paper is to introduce what I feel are the basic ingredients for a new scientific language for formulating research questions on meaningful learning. This so-called Set-Function Language (SFL) is precise and seems particularly well suited for dealing with mathematics, my own area of concern, and science, but it undoubtedly can be used with other subject matters as well. Rather than try to detail the SFL or to summarize the related research that we have completed or have under way, let me simply try to convey the general idea. In the process, SFL and S-R formulations of several meaningful learning tasks will be contrasted.

The SFL is behavioristic, as is the S-R language, but, unlike the S-R language, the SFL denies the primacy of the S-R association. To illustrate some of the advantages of the SFL, consider the situation depicted in Figure 1. Suppose an experimental subject is required to learn to say the appropriate

FIGURE 1


LEARNING
BLACK
LARGE
WHITE
SMALL
TEST ONE
TEST TWO

word when shown a learning stimulus--for example, to say "black" when shown the large black triangle. After the original four S-R pairs are mastered so that the subject can reliably give the correct response to each stimulus, the question remains as to just what was learned. Did the subject learn four distinct pairs--four discrete associations--and notice no relationships between them? Or, did he learn the two principles, "If triangle, then color," and "If circle, then size"?

This question first began to bother me during the summer of 1962. In a study designed by Greeno[3] we found, in a verbal concept learning situation, that essentially a subject either gives the correct response the first time he sees a transfer stimulus or the transfer item is learned in the same way as its control.

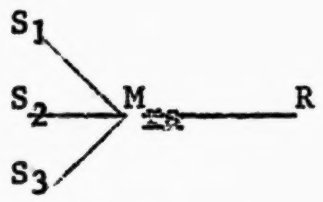
The thought later occurred to me that if transfer obtains on the first trial (if at all),then responses to additional transfer items, at least under certain conditions, should be contingent on the response given to the first transfer stimulus. In effect, a first transfer stimulus could serve as a test to determine what had been learned during the original learning, thereby making it possible to predict what response a subject would give to a second transfer stimulus. To test this assumption, I had a total of about fifteen (highly educated) subjects overlearn the list shown in Figure 1. Prior to learning the list, both the subjects and the experimenter agreed on the relevant dimensions and values--size (large-small), color (black-white), and shape (circle-triangle). The subjects were told to learn the pairs as efficiently as they could, since this might make it possible for them to respond appropriately to the transfer stimuli. After learning, the test 1 stimuli were presented, and the subjects were instructed to respond on the basis of what they had just learned. They were told they were correct no matter what the response. Then, the test 2 stimuli were presented in the same manner.

The results were clear-cut. All but three of these subjects gave the response "black" and "large," respectively, to the test 1 stimuli, and also responded with "white and "small" to the test 2 stimuli. It would appear that when a subject thinks he is right and the new situation remains relevant, he will continue to respond in a similar manner.

On what basis could this happen? It was surely not a simple case of stimulus generalization; the responses were distinct and, in any case, did not depend solely on common stimulus properties. The first test 1 stimulus, for example, is as much like the fourth learning stimulus as the first--and yet, "black" was invariably given as the response rather than "small." Perhaps the simplest interpretation of the obtained results is that most of the subjects discovered the two underlying principles while learning the original list, and later applied them to the test stimuli. In effect, the relationships between the S-R pairs themselves, combined with a response consistency hypothesis, provided a basis for assessing what was learned.

Before introducing the SFL, let us ask how the S-R theorist might represent what was learned in the preceding exercise. In S-R psychology, the basic building block is the association, which was originally viewed as a learned connection between an observable stimulus--light, nonsense syllable, or mathematical problem--and an observable consequence or response--salivation, another nonsense syllable, or solution. A connection or association is said to have been formed if the corresponding response appears with a positive probability whenever the stimulus is presented. Learning a concept, presumably a more complex form of learning, involves the ability to give a common response to any one of a set of stimuli. To say that a subject has acquired the concept of red, for example, implies that the subject is able to say some common response when shown any red object but will not give this response to any non-red stimulus. In short, whereas an association pairs one stimulus with one response, a concept is a many-one relationship.

Since the association is felt to be basic, the S-R theorist has felt obliged to represent the many-one concept relationships as a composite of one-one relationships. This has been made possible by the postulation of mediating links or associations. Thus, the many-one relationship may be represented,

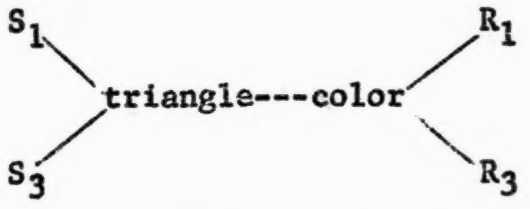


where the stimuli S_1,S_2, and S_3 are connected to the mediating response, $M_{\underline{rs}}$[4] whose stimulus properties, in turn, elicit the observable response, R. In the case of the concept red, $M_{\underline{rs}}$ might be an internalized representation of the label "red."

But the original task we were confronted with involved principles--and principles, from the standpoint of S-R associationism are more complex than either associations or concepts. It is probably because of this felt complexity that most psychologists, particularly experimental psychologists, have simply not been much concerned with principles. Since the notion did not readily fit into their scheme of things, they did what any thinking man would do--ignored the idea.

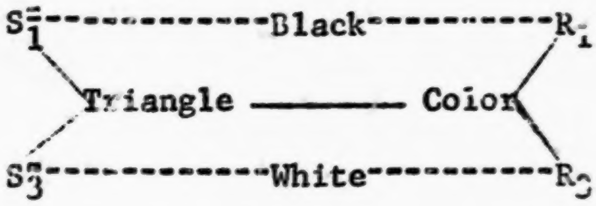
Nonetheless, an increasing number of pedagogically oriented psychologists have come to recognize the central role of the principle in meaningful learning, and at least two association-based representations of the principle have been proposed. Rather than delve into the possible limitations of these formulations, I shall propose an S-R formulation of my own--and then attack it.

First, consider the representation,



This representation has been chosen to reflect the principle,"If triangle, then color," as it relates to the situation depicted in Figure 1. In this case, the overt stimuli, S_1 and S_3 , are presumed to elicit the mediator "triangle" which, in turn, elicits the mediator "color." "Color," then, is presumed to elicit R_1 and R_3--with equal probability. Of course, we know that this is not what happens from the study just described; S_1 goes with R_1 , and S_3 with R_3.

To overcome this inadequacy, we may postulate a second pair of connecting chains,



According to this interpretation, S_1 elicits R_1 and not R_3 because S_1 elicits "black" as well as "triangle" and has no direct relationship to "white." Of course, S_3 elicits "white" for the same (implied) reason. With this representation in hand, the S-R associationist is now able to predict the results of the experiment described. "Black" is given as the response to the first test stimulus, a small black triangle, because of the prelearned association between the stimulus and "black" and the learned mediating association, "triangle elicits color." In short, two associative connections are better than one.

Rather than go into the SFL representation of principle learning at this point, let me first add more fuel to the fire by presenting another example. Suffice it to say at this point that the essence of a principle is captured reasonably well by a statement of the form, "If A, then B.[5]" Consider the S-R pairs,

$(7, 1, 6, 4) \longrightarrow 9$

$(4, 8, 9, 3) \longrightarrow 10$

$(6, 5, 8, 9) \longrightarrow 5$

$(9,7,8,6) \longrightarrow$

$(5,1,8,3) \longrightarrow$

Suppose the learner is posed with the task of determining, from the instances on the left, that principle which will allow him to predict the appropriate responses to the stimuli on the right. He might find that a difficult task.

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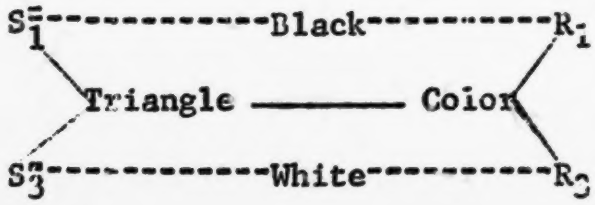

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(9,7,8,6) ⟶
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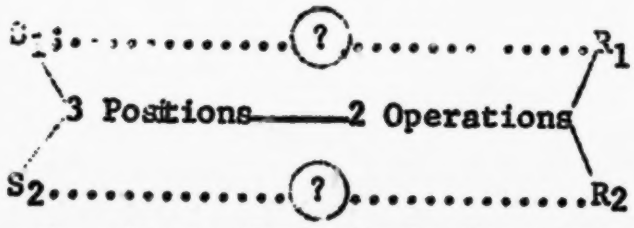
Suppose the learner is posed with the task of determining, from the instances on the left, that principle which will allow him to predict the appropriate responses to the stimuli on the right. He might find that a difficult task.

But, if he is told that the responses (integers) can be derived uniquely from the integers in the first, third, and fourth positions of the stimuli, it might be easier. It would undoubtedly be still easier, if he is told that the operations of addition and subtraction are involved.

More is involved in this example than in the first. The learner is required not only to discover the underlying principle, and a more complex principle at that, but hints have been given as to how he might accomplish the task.

How might these contingencies be formulated in terms of associations? An immediate thought is that the position and operation cues might be viewed as mediating links,



But, as a quick look will make clear, that simply does not work. Further elaboration would be necessary to indicate why S_1 goes with R_1 , and S_2 with R_2. Of course, there undoubtedly are other alternatives, but I think you will agree that any S-R representation of these continencies is likely to be extremely complex.

It is important to emphasize that this situation was not picked arbitrarily simply to embarrass S-R psychologists. Stimulus dimensions (e.g., color, position), which uniquely determine the responses and combining operations (e.g., name the color, addition and subtraction), by which the responses are derived from these stimulus properties, appear to be crucial aspects of *all* principles.

Fortunately, these characteristics play a central role in the SFL. In fact, it is assumed that four elements, *I*, *D*, *O*, *R*, are needed to specify a principle. The stimulus properties in the set, *I*, tell when the principle is to be used; those in *D* tell which properties determine the response; and the combining operation, *O*, tells how the response properties, *R*, may be derived from those in *D*.[6] Obviously, the attribute and operation cues, involved in the above situation, find natural counterparts in this characterization. Thus, *D* involves the three dimensions corresponding to the three positions, and O is a composite operation involving addition followed by subtraction. In addition, the response set R involves the integers variable (i.e., set of integer responses), while those stimulus properties (in I) which determine when the principle is to be applied are those associated with being a "four-tuple" of integers. Notice that the properties in *I* involve a first, third, and fourth position (those properties associated with *D*), so that *I* necessarily involves *D*. In the first of the (two) task *I* principles, *I* involved colored triangle, *D* color, *O* color naming, and *R* the color names.

Since principles, as well as associations and concepts, can be represented in the S-R language, it is appropriate to ask whether these notions can also be represented in the SFL, in which the *principle is taken to be basic*. As can be seen in Table 1 this is indeed the case. Furthermore, unlike the S-R language, explicit distinctions are made between: (1) the observable S-R instances of a principle--the denotation; (2) the principle itself, that which underlies the behavior and whose presence can be inferred only indirectly; and (3) statements of the principle in symbolic form.

The denotation of a principle is simply taken to be equivalent to the mathematical notion of the function, a notion which may be defined as a set of ordered S-R pairs such that to each functionally distinct stimulus there is one corresponding functionally distinct response. The denotation of a concept is simply represented as a function in which there is one response common to all stimuli. To represent an ordinary association, the set is further restricted so as to include only one S-R pair.

The principle itself is characterized as an ordered four-tuple (*I*, *D*, *O*, *R*). where *I*, *D*, *O*, and *R* are as previously defined. In the case of concepts and associations, there are certain restrictions placed on these elements, but they need not concern us here. Finally, notice that the statements in the column on the right are constructed from symbols *I'*, *D'*, *O'*, and *R'*, representing the constructs *I*, *D*, *O*, and *R*.

Table 1

SFL Representations of the Association, Concept, and Principle

Denotation (Observables)	Principle Underlying Performance (Inferred)	Principle Statement (Observable)
Principle		
$\{(S_j, R_j) \mid j \in J\}$	(I, D, O, R)	If I', then $R' = O'(D')$
Concept		
$\{(S_j, R) \mid j \in J\}$ (Note: One response)	(I, D, O, R)	If I', then $R' = O'(D')$
Association		
$\{(S, R)\}$ (Note: One pair)	(I, D, O, R)	If I', then $R' = O'(D')$

Due to space limitations, I will not go into figure 2 in any detail. Let me simply point out that while there are common relationships between different pairs of the experimental paired-associate (PA) list, none exists in the control list. The principles involved in the experimental list might be stated, "If black, then shape," and "If white, then size." "If a small black circle, then circle," is a candidate for one of the control list principles.

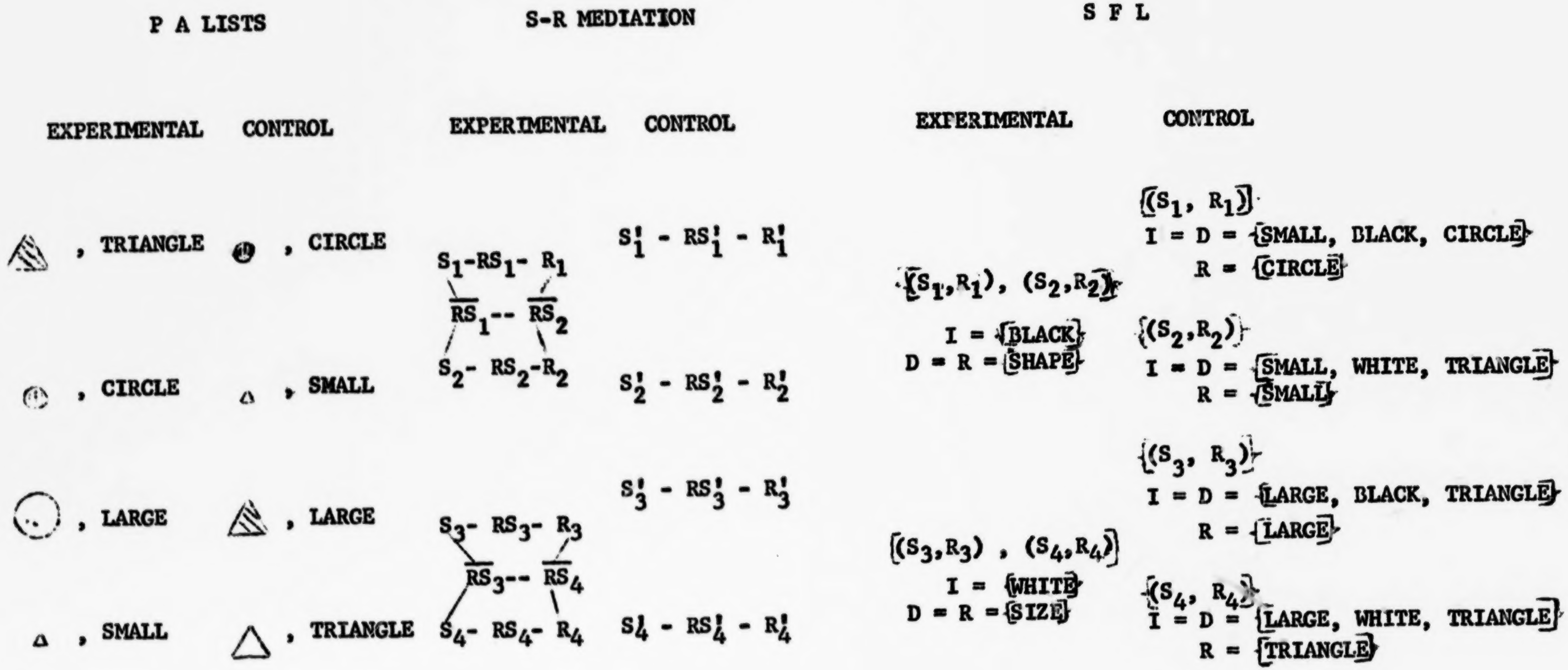
Space limitations also demand that I not attempt to detail my reasons for preferring the SFL to the S-R language. I would, however, like to mention three reasons why the SFL appears to me to be better suited for formulating research on meaningful learning, in general, and mathematics and science learning, in particular.

The first major reason is that principles are so critical, even in the simplest forms of meaningful learning. And, as we have seen, it is much simpler to represent what is learned in the SFL than it is in terms of more cumbersome S-R representations.

My second reason for preferring the SFL is that meaningful learning involves the ability to make the appropriate response in a class of responses to any one of a class of stimuli. Learning single principles may make such performance possible but not single associations. Lest there be some confusion, the stimuli and responses to which I am referring may be quite distinctive and not merely differ slightly along some physical dimension (i.e., the stimuli and responses may be functionally distinct). In simple learning, where the stimulus dimensions are continuous, the ability to give a new response to a new stimulus had traditionally been attributed to S-R generalization. Some of our recent data[7], however, suggests that such a postulate may be inappropriate in theories of meaningful learning. Meaningful transfer may more easily and accurately be accounted for in terms of what principle is learned.

The third and perhaps most important reason is simply that there are many situations in meaningful learning--we have reviewed just two--which, while easily

FIGURE 2


P A LISTS
S-R MEDIATION
S F L
EXPERIMENTAL
CONTROL
EXPERIMENTAL
CONTROL
EXPERIMENTAL
CONTROL
, TRIANGLE
, CIRCLE
, CIRCLE
, SMALL
, LARGE
, LARGE
, SMALL
, TRIANGLE
S_1-RS_1- R_1
$\overline{RS}_1$-- $\overline{RS}_2$
S_2- RS_2-R_2
S_3- RS_3- R_3
$\overline{RS}_3$-- $\overline{RS}_4$
S_4- RS_4- R_4
S_1' - RS_1' - R_1'
S_2' - RS_2' - R_2'
S_3' - RS_3' - R_3'
S_4' - RS_4' - R_4'
$\{(S_1,R_1), (S_2,R_2)\}$
I = {BLACK}
D = R = {SHAPE}
$\{(S_3,R_3), (S_4,R_4)\}$
I = {WHITE}
D = R = {SIZE}
$\{(S_1, R_1)\}$
I = D = {SMALL, BLACK, CIRCLE}
R = {CIRCLE}
$\{(S_2,R_2)\}$
I = D = {SMALL, WHITE, TRIANGLE}
R = {SMALL}
$\{(S_3, R_3)\}$
I = D = {LARGE, BLACK, TRIANGLE}
R = {LARGE}
$\{(S_4, R_4)\}$
I = D = {LARGE, WHITE, TRIANGLE}
R = {TRIANGLE}

represented in SFL terms, are difficult, if not impossible, to formulate in the S-R language.

There are other situations which pose even more difficulty for the S-R language, while yielding to SFL analyses. During the past year or so we have either completed or have SFL-based research under way on the following topics: (1) the role of attribute (D) and operation (O) cuing in learning mathematical principles, (2) rule generality in mathematics learning, (3) the role of symbolism in mathematics learning, and (4) the expository presentation of what mathematical strategy is learned in discovery learning. In the future, we hope to get into even more complex cognitive tasks such as: (1) identifying (and representing) heuristics of use in mathematical problem solving and proving mathematical theorems, (2) extending the SFL so as to provide a rigorous basis for such things as task analysis and Bloom's taxonomy of educational objectives (cognitive domain), and (3) the construction of instructional sequences to meet multiple objectives. So far it appears that the SFL is capable of representing, if not all, then certainly many of the critical characteristics of such situations.

In the final analysis, the choice between scientific languages (and theories) involves efficiency and cohesiveness as well as the sheer ability to represent or account for observable phenomena. It is in this sense that the much heralded adaptive quality of the S-R language too often has led psychologists to overlook the fact that it is always possible to patch up an existing formulation to meet new situations. Parsimony does not simply refer to the maintenance of existing concepts but to the formulation of emerging structures in the simplest possible way. I might add, in conclusion, that what is presently being done with the S-R language (e.g., the incorporation of hierarahies and motivation factors) is quite analogous to what pre-Copernican astronomers were doing when they invented epicycles to represent planetary motions in an attempt to salvage geocentric theory.

Having made that emotion-laden comment, I rest my case.

NOTES

1. This article is based on a paper read at the American Educational Research Association convention in Chicago on February 19, 1966. A description of some of the recent empirical research, based on the Set-Function Language, may be found in my "Precision in Research on Mathematics Learning," Journal of Research in Science Teaching, IV(December, 1966), 253-274.

2. F.J. McDonald, "Meaningful Learning and Retention: Task and Method Variables," Review of Educational Research, XXXIV(December, 1964), 542.

3. J.G. Greeno and J.M. Scandura, "All-or-None Transfer Based on Verbally Mediated Concepts," Journal of Mathematical Psychology, III,(July,1966) 388-411.

4. Typically, a distinction is made between a mediating response, M_r, and its stimulus properties, M_s, so that the chain $S \rightarrow M_r \rightarrow M_s \rightarrow R$ would be used rather than $S \rightarrow M_{rs} \rightarrow R$. The shorter form suffices for present purposes.

5. This representation was introduced by R.M. Gagne (e.g., "The Conditions of Learning" New York: Holt, Rinehart & Winston, 1965).

6. More recently, I have found it useful to make a distinction between rules and principles, a rule being completely characterized by D, O, and R. The interested reader is referred to the references in footnotes 1 and 7.

7. In J. M. Scandura, E. Woodward, and F. Lee, "Rule Generality in Mathematics Learning," American Educational Research Journal, in press, no S-R generalization gradient was found in learning mathematical rules. Such a result runs contrary to what would be expected by Most S-R theorists on the basis of experiments on simple learning.

The Mathematics Teacher

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"CONCEPT" LEARNING IN MATHEMATICS*

(Research in the Emerging Discipline of Psycho-Mathematics)

Joseph M. Scandura

Doing basic research on mathematics learning is a risky business. In the first place, it is hard to know whether one is asking the right questions and, in the second place, it has been difficult to formulate significant questions involving mathematics learning in a researchable form. One of the major reasons for this state of affairs has been the lack of any suitable theoretical superstructure from which to work. The purpose of this paper is to outline some of the theoretical work underway at the University of Pennsylvania and to show how this theoretical work has helped to improve our understanding of how mathematics is learned.

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*This paper is based on invited talks given during the Spring of 1967 at the University of Delaware and the State Supervisors of Mathematics Meeting and the NCTM convention in Las Vegas. The author would like to thank Joan Bracker for her assistance in the preparation of this article. Her participation was made possible by a graduate research training fellowship supported by the U.S. Office of Education.

Background

Theoretical development in educational psychology has been extremely slow. One major reason has been the typically imprecise definition of independent and dependent variables in research on meaningful learning and teaching. Such research can bear only an ambiguous relationship to theory. Similarities and essential differences often go undetected. Stating research objectives and defining variables in unambiguous terms, however, is not sufficient. The teaching-learning process has all too frequently been studied in terms of such "administrative" variables as class size, grade level, I.Q., and amount of teaching experience. To be theoretically relevant, the variables chosen must have broad explanatory potential. They should not merely be symtomatic of and inextricably related to the question at hand. Theory development depends on much more than mere fact finding.

To provide a substantive base for their research, educational psychologists have frequently resorted to the languages, paradigms, and theories of the mother science of psychology. Mediational elaborations and operant conditioning paradigms of the S-R language and more general, but less well specified, cognitive theories have been popular.

Each approach has important limitations.[1] From one point of view, parsimony suggests that the properties of overt S-R associations should also be attributed to mediational links. Yet, practice has shown that mediational interpretations become increasingly cumbersome and less precise as situations become more complex. Similar difficulties have plagued researchers who have used operant techniques to study meaningful verbal learning. The results simply are nowhere near as clear in complex human learning as they are in the "Skinner Box." It is increasingly recognized, for example, that knowledge of results is not directly analogous to feeding a pigeon and that, in any case, other factors, such as subject matter structure, are probably of greater importance in promoting efficient learning. A general limitation of cognitive theories is their relative imprecision. Typically, "cognitions" are either not clearly specified in observable terms or are only partially defined. Under these conditions it has been impossible to construct a predictive theory--the sort of theory needed if practical implications are to be obtained.

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[1] Joseph M. Scandura, "The Basic Unit in Meaningful Learning--Association or Principle?", The School Review, (1967, in press).

Joseph M. Scandura, "Precision in Research on Mathematics Learning (The Emerging Field of Psycho-Mathematics)." Journal of Research in Science Teaching, (1967).

Joseph M. Scandura, The Mathematical Formulation of Educational Research Questions, (Washington: U.S. Office of Education (Project 6-8002), 1967).

In short, the choice to date has been between a precise, but seemingly inappropriate S-R language, and presumably more relevant cognitive formulations which leave much to be desired insofar as scientific cohesiveness and rigor are concerned.

In an attempt to overcome these difficulties, I have proposed a new scientific language for formulating research questions on meaningful learning. Because the language is framed in terms of the mathematical notions of sets and functions, the name Set-Function Language (SFL) was adopted.[2]

I won't go into detail here, but before outlining the SFL it may be instructive to briefly consider the S-R mediation language. In S-R psychology, the basic building block is the association, a construct which was abstracted directly from observed connections between overt stimuli and overt responses.

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[2] See footnote 1. The descriptor "Set-Function" should not be confused with set functions.

Learning a concept, presumably a more complex form of learning, involves the ability to give a common response to any one of a set of stimuli. To say that a subject has acquired the concept of "red," for example, implies that he is able to give some common response, when shown any red stimulus object, but will know not to give this response to any non-red stimulus. Similarly, a child may be said to have acquired the concept of "four" if he can say "four" when presented with any conglomeration of four objects but will not say four to any conglomeration not containing four objects--i.e., assuming, of course, that the child is operating under the same set of instructions in each case. In short, whereas an association pairs one stimulus with one response, a concept is a many-to-one relationship.[3]

S-R theorists have felt obliged to represent the many-to-one concept relationship as a composite of one-to-one associations,

$$\begin{tikzcd}
S_1 \arrow[dr, dash] & & \\
S_2 \arrow[r, dash] & M_{rs} \arrow[r, dash] & R \\
S_3 \arrow[ur, dash] & &
\end{tikzcd}$$

Notice that the stimuli S_1, S_2, and S_3 are connected to the mediating response M_{rs} whose stimulus properties, in turn, elicit the observable response R.

The Basic Unit in Meaningful Learning

Most subject matter learning involves neither associations nor concepts but, as they have been variously called by different investigators, rules, principles, schemas, heuristics, and TOTE units. This is true even more so of mathematics learning. To be more specific, meaningful learning *implies* the ability to give the appropriate response in a class of responses to any stimulus in a class of stimuli. Unfortunately, this fact has often been overlooked because the term "concept" has been used so widely in discussing subject matters. When we say that a child "has the *concept* of addition," for example, what we probably mean is that he can give the appropriate sum when presented with *any* pair of numbers. Put another way, the learning involved connects a large class of stimuli with a large class of responses. By definition, a concept connects a class of stimuli to exactly *one* response.

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[3] I should like to point out that in talking about single stimuli (responses), I am actually referring to equivalence classes of overt stimuli (responses) in which the members (i.e., the stimuli) of the classes are either indistinguishable or otherwise play exactly the same role. For example, the stimuli "5", "five", "Five", "FIVE" are all equivalent in so far as the number five (as opposed to the numeral "five") is concerned.

To see what is involved in meaningful learning, learn the following S-R pairs (i.e., overt inputs and outputs): (4 3 1) → 3, (8 1 6) → 2, (7 9 2) → 5, and (9 5 1) → 8. Now, on the basis of what you have just learned, give what you think should be the responses to the stimuli (7 2 1) and (4 7 2). Did you give the responses 6 and 2? If so, you probably acquired a unit of knowledge (i.e., rule) which might be stated, "Subtract the number in the third position from that in the first." If you did not give these responses, you presumably learned the pairs on the left as discrete entities--i.e., as distinct associations without noticing any relationships between them. (Of course, I didn't indicate that there was any such relationship so you may not have been looking for one.) In this case, the rule governed responses, 6 and 2, would be expected on the basis of chance alone.

I should like to emphasize that this situation was not picked arbitrarily simply to embarass S-R psychologists. Whereas a "patch job" can be done with certain special cases, I believe that I am safe in saying that to date no satisfactory way of representing rules *et al* solely in terms of associations has been found.[4] Stimulus dimensions which uniquely *determine* the responses (e.g., the first and third positions) and the combining *operation* or transformation by which the responses are derived from the determining stimulus properties (e.g., subtraction) appear to be crucial aspects of *all* rules. While stimulus properties and derived responses play a central role in S-R mediation theory,[5] there is no counterpart for the transform or combining operation.

The Set-Function Language (SFL)

Fortunately, all of these characteristics play a central role in the SFL. In fact, I[6] have proposed that four characteristics are needed to specify a *principle*. Three of these characteristics specify a rule and a fourth determines when the rule is to be applied. Those stimulus properties which *determine* (D) the corresponding responses constitute one such characteristic, the covert *responses* or derived stimulus properties (R) are another, and the transform or combining *operation* (O) by which these covert responses are derived from the determining properties is the third. The fourth consists of those, usually higher order, stimulus properties which *identify* (I) the rule to be applied. For example, the rule, N^2, for summing number series, where N is the number of terms, applies only to those series which consist of the consecutive odd integers beginning with one (e.g., $1 + 3 + 5 + 7 = 4^2 = 16$).

Since principles, as well as associations and concepts, can be represented in the S-R mediation language, it is appropriate to ask whether these notions can also be represented in the SFL, in which rules and principles are taken to be basic. This is indeed the case. Furthermore, unlike the S-R language. explicit distinctions are made between: (1) the observable S-R instances of a rule--the denotation, (2) the rule or principle itself, that which underlies the behavior and whose presence can be inferred only indirectly, and (3) statements of the rule or principle in symbolic form.

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[4] See footnote 1.

[5] They correspond, in fact, to mediating responses and their stimulus properties, respectively.

[6] See footnote 1.

The denotation is simply a _function_, a (mathematical) notion which may be defined as a set of ordered stimulus-response pairs such that to each stimulus there is one corresponding response. The denotation of a concept is simply represented as a constant function in which there is _one_ response common to all stimuli. To represent an ordinary association, the set is further restricted so as to include only one S-R pair.

The rule construct is characterized as an ordered triple (D, O, R), where D, O, and R are as defined above. Principles, of course, are ordered four-tuples (I, D, O, R). In the case of concepts and associations, there are certain relationships among these characteristics,[7] but they need not concern us here. In stating rules or principles, I have used primes to distinguish the signs used to represent the constructs I, D, O, and R from the constructs themselves.

The one point I would like to emphasize is that even S-R theorists have been _forced_ to adopt the idea of a _transformation_ or combining operation in order to represent meaningful learning and thinking. Berlyne[8] has done this, for example, in his fine book _Structure and Direction in Thinking_. In his latest article on the topic of principle learning, Gagne[9] has adopted this point of view as well. Even many long-time mediation enthusiasts have recently become convinced that mediation theory is inappropriate for dealing with verbal learning.[10]

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[7] For more details see Joseph M. Scandura, _The Mathematical Formulation of Educational Research Questions_, (Washington: U. S. Office of Education (Project 6-8002), 1967).

[8] David E. Berlyne, _Structure and Direction in Thinking_ (New York: Wiley, 1965).

[9] Robert M. Gagne, "The Learning of Principles," in H. J. Klausmeier and C. W. Harris _Analysis of Concept Learning_ (New York: Academic Press, 1966), pp. 81-96.

In so doing, these theorists are in fact _denying_ the primacy of the association as a construct. The idea of a transformation or combining operation takes over this primary role. I might note parenthetically that what here have been called response determining properties of stimuli, S-R theorists have called mediating responses. Similarly, our covert responses (or, derived stimulus properties) correspond to stimulus properties of mediating responses. _What the combining operation does is to make explicit how the mediating stimuli are determined from the antecedent mediating responses._ While this process may not be important in many forms of simple learning, we have just seen here how it becomes _crucial_ in rule learning. The transformation, mapping, combining operation, or whatever term is used, becomes of central concern, and to still call any suitable representation associationistic or even neo-associationistic is stretching the term beyond its reasonable limits.

In the final analysis, of course, the choice between scientific languages (and theories) involves efficiency and cohesiveness as well as the sheer ability to represent or account for observable phenomena. It is in this sense, that the much heralded adaptive quality of the S-R language too often has led psychologists to overlook the fact that it is _always_ possible to "patch-up" an existing formulation to meet new situations. Parsimony does not simply refer to the maintenance of existing concepts but to the formulation of emerging structures in the simplest possible way. I might add that what is presently being done with the S-R language (e.g., the inclusion of associative structures, reference mechanisms, etc.) is quite analogous to what pre-Copernican astronomers were doing when they invented epicycles to represent planetary motions in an attempt to salvage geo-centric theory.

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[10] Personal communication.

EMPIRICAL RESEARCH BASED ON THE SFL

Let me turn now to a consideration of two problems that have long plagued mathematics educators, the problems of rule-generality and discovery learning. In the space remaining, I shall attempt at least partial resolutions of these two problems and, in the process, will indicate how the SFL helped to achieve these resolutions by making it possible to formulate the underlying questions in a precise way. Ernest Woodward and Frank Lee assisted me with the Rule-Generality Study and the Discovery Learning Study was conducted with now Dr. William Roughead.

Rule Generality[11]

In instructional situations, the question often arises as to how general the presentation of material ought to be. Subject matter specialists and most educators tend to emphasize that the more general the presentation, the more useful it will be. Learning oriented experimental psychologists, on the other hand, are often inclined to point out that the more specific the presentation, the better the learning.

The question of generality arises particularly often in mathematics instruction. Should addition and subtraction be taught as two distinct, although related, operations, as has been done traditionally, or as one operation as is done in more modern treatments? Should the three cases of percentage be taught separately or as variants of the rule, "base X rate = percentage?" Should pupils be taught the method of "casting out nines" or be taught the more general principles of modular arithmetic? How generally should theorems be stated? Proofs? The answer to these and related questions hinges, in part, on the learnability and utility or scope of the rules involved.

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[11] This section is based on Joseph M. Scandura, Ernest Woodward, and Frank Lee, "Rule Generality and Consistency in Mathematics Learning," American Educational Research Journal, (1967, in press).

Mathematics educators have concerned themselves with such questions, but they have had to make judgments on largely intuitive grounds. There is a real need to better understand the psychological principles involved. Unfortunately, however, previous studies involving rule learning have dealt only incidentally with this problem. Perhaps more important, even the best designed studies in this area are subject to criticism for failure to distinguish between structure and behavior variables. The variables chosen (e.g., rule and example given, discovery, answer given) are often merely symptomatic of, rather than basic to, what is involved.

The fundamental assumption underlying our approach to the problem was that more rapid progress can be made by distinguishing clearly between structure and behavior variables and by identifying the important parameters of each. I would go even farther and say that we can never hope to understand mathematics and other subject matter learning without making such a distinction.

At the time the generality study was designed, the SFL had only developed to the point where rules were defined in terms of their denotative sets of ordered stimulus-response pairs. No consideration was given to the nature of the underlying rule construct. Even so, the fact that sets can be ordered as to their inclusiveness led naturally to the question of rule generality. Not only did this question have practical relevance but, more important from the standpoint of theory, the SFL provided a basis for rigorously defining just what is meant by generality. One rule is said to be more general than another if the denotative set of the former includes that of the latter as a proper subset (i.e., the former set includes all of the instances of the latter plus some of its own).

Our primary motivation for the rule generality study, then, was to "try out" this definition, to see if it had the sort of straight forward behavioral implications we had hoped. In particular, notice that all S-R instances of a principle are treated equally. Any stimulus within the scope of a rule should provide an adequate test of its acquisition. Similarly, performance on extra-scope test stimuli should be uniformly nil. Once learned, a highly general rule would, of course, be expected to induce appropriate performance on a wide variety of tasks. At the same time, however, it is quite possible that the ease of learning a rule statement, as judged by the ability to use it, may vary directly with its specificity.

Another facet of our rule generality research concerned the consistency with which a learned rule is applied. In my earlier research,[12] it was found that, under certain conditions, experimental subjects respond consistently in accordance with a derived rule. When told that their first response was correct, those subjects who used a rule as the basis for responding to a first test item also used the rule on a second test item. These pilot results were obtained in a discovery-learning situation with simple materials. There was a need to extend this finding to more complex subject matters which are presented by exposition. In general, we found strong support for this contention. Apparently, people tend to respond in a consistent manner unless the context is changed or feedback otherwise indicates that the rules have changed.

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[12] e.g., See Joseph M. Scandura, "Teaching-Technology or Theory," American Educational Research Journal, 1966, 3, 139-146.

For the purposes of this discussion, we shall be primarily concerned with only two hypotheses. First, performance on problems within the scope of a rule does not differ appreciably and successful problem solving is limited almost exclusively to within-scope problems. Second, the ease of learning a rule statement so that it can be applied to within-scope problems varies directly with the rule's specificity. Thus, the more general a rule statement, the poorer the learning.

To test these hypotheses we conducted two experiments, only one of which I shall outline here. The crux of the experimental design and the results can be seen in Table I. The three groups of experimental subjects (Ss) were undergraduates enrolled in a mathematics education course for elementary teachers. Each group was presented with one of three rule statements of varying generality. All of the Ss were then tested on the same three problems. Rule S was the most specific and was appropriate for solving any problem in the certain class--problem one was one such problem. Rule SG was more general and was potentially applicable to a wider range of problems. In particular, it was logically sufficient to solve both problems one and two, but not problem three. Rule G was the most general rule; it was applicable to all three test problems.

TABLE I.

NUMBER OF CORRECT SOLUTIONS

	N	Problem One	Problem Two	Problem Three
Group S	17	13	0	1
Group SG	17	5	4	0
Group G	17	5	5	4

In agreement with the first hypothesis, there was almost no extra-scope transfer and performance on within-scope problems was essentially the same. Notice, in particular, that the performance obtained could be predicted on the basis of a strictly logical argument. If people learn an unfamiliar rule at all then they should be able to apply it to any problem or stimulus within the scope of the rule but not to similar problems beyond the scope of the rule.[13]

Defining rule generality in terms of the scope of the respective denotative sets of S-R pairs, however, does not provide a sufficient basis for explaining the results pertaining to the second hypothesis which was concerned with the learnability of rule statements of different generality. In the beginning, this hypothesis was based simply on intuition.

To check this hypothesis, consider the performance of the three groups on problem one. Since this was the only problem within the scope of all three rules, we expect on the basis of our hypothesis that group S would do better than group SG which, in turn, would do better than group G. As you can see, group S did do much better than the others, but the performance of the subjects in groups SG and G did not differ appreciably.

To explain these results, it was found necessary to define an underlying construct, a cognitive competency underlying the behavior we observed. This led me to the four-tuple characterization previously outlined. When the competencies necessary for interpreting the rule statements of varying generality were analyzed it became apparent that the more general the rule, the more is required of the learner. Thus, to apply a highly general rule statement, once memorized, requires that the learner be able to apply any rule of lesser generality but not conversely. To subtract any two numbers, for example, implies that 2 can be subtracted from 6, but being able to subtract 2 from 6 does not imply that the learner can subtract with any two numbers.

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[13] Actually, the facts are not quite this simple, but the statement appears to be a good first approximation. John Durnin and I have a study underway in the Penn Laboratory which we hope will add further clarification.

The fact that performance of the groups given the most general (G) and intermediate (SG) rule statements did not differ can be attributed to prior learning. In effect, those additional abilities needed to interpret and, hence, to apply rule G were not likely to have caused the college students involved any difficulty.

Since postdiction is held in generally low regard in the psychological (but not scientific) community, let me add a brief word of defense. In this particular study, we began with a precise definition of rule generality based on a preliminary version of the SFL; we conducted a study, and, finally, we used the results to extend and otherwise improve the very foundations on which our a priori analysis had been based.

What educational implications can be drawn from this study? To begin with, the results of these experiments demonstrate, in a rather conclusive fashion, the behavioral relevance of rule generality. For the most part, successful performance was noted only on tasks within the scope of verbally stated rules. When rules are presented in an expository fashion, it is normally too much to expect generalization to problems to which the principle does not immediately apply.[14]

Potentially of even greater practical significance was the lack (there was one exception) of performance differences on the within-scope problems and the consistency results cited earlier. The former result demonstrates that, under certain specifiable conditions, any stimulus within the scope of a rule is equally as difficult to respond to correctly as any other. Furthermore, the obtained consistency results suggest that only one (new) test stimulus is needed to determine whether, in fact, a given rule has been learned. Under certain specifiable conditions, no more information is gained by using additional test instances. These results could have far-reaching implications for the development of highly efficient measuring instruments.

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[14] See footnote 13. I might add that some transfer does take place and we are now attempting to pin down the source of this transfer.

In addition, the pronounced tendency of the Ss to attack all of the test problems in the same way, irrespective of whether the procedure used was appropriate, suggests that knowing how to solve problems and knowing when to use this knowledge are quite distinct. Testing for the latter ability necessarily must involve the presentation of extra-scope problems.

Most important, in view of the problem I posed originally, this study helps to reconcile the views of subject matter and learning specialists as to how general the presentation of material ought to be. While one may expect maximum transfer potential by introducing rules of the greatest possible generality, this transfer potential is bought at a price the teacher may or may not be willing to pay. The greater its generality, the harder a rule statement is to interpret. Hence, before determining how general a rule statement should be, it is essential to first consider whether the students involved have the necessary requisite interpretative abilities.[15]

Actually many teachers already do this, at least intuitively. All that the present study does is to make these intuitions more explicit. Frankly, I am always pleased when our results conform to what might be called "common sense." Too often psychological results, while perhaps relevant to animal learning or the memorization of nonsense-syllable lists, have very little to say about the learning of mathematics and other subject matters.

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[15] Joseph M. Scandura, "On Ability Treatment Interactions (The Learning and Interpretability of Symbolically and Verbally Stated Mathematical Rules)," Journal of Educational Psychology (1967, in press).

Discovery Learning[16]

One of the fundamental assumptions underlying several of the new mathematics programs is that discovery methods of teaching and learning increase the student's ability to learn new mathematics. Indeed, this assumption has guided the development of many new curricula in all of the subject matter fields. Attempts to demonstrate advantages or disadvantages of self-discovery, however, have either failed, been open to criticism on scientific grounds, or are seemingly inconsistent even when apparently well-controlled.

Research on discovery learning has been confounded by differences in terminology, the frequent use of multiple dependent measures, and vagueness as to what is being taught and discovered. While the difficulties due to the use of inconsistent terminology can often be minimized by a careful reading of research reports, the use of multiple dependent measures often makes it impossible to unambiguously interpret experimental results. Several investigators, for example, have found that groups which are given an expository statement of a rule perform better on transfer tests than groups which are required to discover this rule for themselves from instances of the rule. The obtained differences in transfer ability, however, may well have been because the discovery groups simply did not discover the rule.

Gagne and Brown[17] overcame the dependent measure problem by equating original learning and investigating only transfer differences on new problems. On the basis of an analysis of the learning programs used in the Gagne and Brown study, Eldredge [18] hypothesized that the obtained results could have been due to a number of flaws in the programs used. Eldredge proposed that exposition and discovery situations may be better characterized as differences in order of presentation. Exposition may be defined as giving rules and then examples of these rules, whereas discovery may be defined as giving the examples and then the rules. Contrary to his hypothesis, however, his discovery group did evidence more transfer than his exposition group. Unfortunately, there were a number of difficulties with the study that make the results difficult to interpret.

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[16] William G. Roughead and Joseph M. Scandura, "Sequencing and the Expository Presentation of What is Learned in Mathematical Discovery," (paper read at the A.E.R.A. convention in New York on February 18, 1967).

[17] Robert M. Gagne and L. T. Brown, "Some Factors in the Programming of Conceptual Learning," Journal of Experimental Psychology (Vol. 62, 1961), pp. 313-321.

The Set-Function Language was used as an aid in removing these difficulties. The resulting analysis of what is involved in discovering rules indicates that discovery learners learn "something" by which they can derive solutions to an entire class of problems. Roughead and I called this "something" a derivation rule. Thus, discovery learners who actually succeed in making a discovery, should be expected to perform better than expository learners on tasks which are within the scope of such a derivation rule. If the new problems presented have solutions beyond the scope of a discovered derivation rule, however, there would be no reason to expect discovery Ss to have any special advantage.

This study was concerned with two basic questions. First, can "what is learned" by discovery be identified and if so, can that knowledge be taught by exposition with equivalent results? According to the SFL, all behavior is controlled by rules so that there might well be some identifiable rule which is equivalent to "what is learned" by discovery. Specifically, we hypothesized that "what is learned" by guided discovery in the Gagne and Brown study could be identified and, hence, could be presented by exposition. The second question was, how is "what is learned" by discovery dependent on what the learner already knows and/or the nature of the discovery treatment itself? More particularly, we hypothesized that the discovery of a derivation rule can actually be hindered by having too much prior information.

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[18] Garth M. Eldredge, "Discovery vs. Expository Sequencing in a Programmed Unit on Summing Number Series," in Gabriel M. Della-Piana, Garth M. Eldredge, and Blaine R. Worthen, Sequence Characteristics of Text Materials and Transfer of Learning (Salt Lake City: Bureau of Education Research, 1965).

Assuming transfer depends only on whether or not the derivation rule is learned, sequence of presentation should have no effect on transfer so long as the subject is forced to learn the underlying derivation rule. That is, presenting the derivation rule by exposition or by guided discovery either before or after presenting the desired responses should have no effect on performance on transfer tasks. On the other hand, if a discovery program simply provides an opportunity to discover (with hints as to the solution) but does not guide the learner through the derivation procedure, sequence of presentation might well have a large effect on transfer. Assuming the learner is capable and motivated, he may well succeed in determining the appropriate responses and, in the process, discover a derivation rule. It is not likely, however, that a person would learn such a derivation rule if he already knew the correct responses.

We made three hypotheses: 1) what is learned by guided discovery can be presented by exposition with equivalent results; 2) presentation order is not critical when learners are effectively "forced" to learn derivation rules, either by exposition or by guided discovery; 3) presentation order is critical when the discovery guidance provided is specific to the respective responses sought rather than relevant to a general strategy or derivation rule.

The task we used was essentially identical to that used by Gagne and Brown, and Eldredge and involved finding formulas for summing the terms in number sequences. That is, the stimuli were number series, like 1 + 3 + 5 + 7, and the responses were formulas in n, the number of terms, for summing such series. For example, the appropriate formula for summing $1 + 3 + 5 + 7 + \ldots + 2n-1$ is n^2.

Using the SFL as a guide (i.e., by identifying, in turn, D, O, and R) we were able to identify that derivation rule taught in the guided discovery program used by Gagne and Brown. On the basis of this knowledge, four programs were constructed: 1) the formula-given program simply stated the correct summing formula for each problem series confronted in the learning program; 2) the guided discovery program remained essentially as it was in the earlier studies; 3) the expository program consisted of a precise expository description of that derivation rule which was presumably equivalent to that learned by guided discovery. It consisted of a general procedure by which the desired formulas could be derived; 4) in the opportunity-to-discover program, the problem sequences were presented along with encouragement and hints as to what the desired formulas were. These hints involved such statements as "the formula has a '2' in it." The same number sequences were used in each of these four programs.

Seven treatments were constructed by combining these four basic programs. After going through a common introductory program, one group of subjects simply went through the formula program. The other six groups received both the formula program together with one of the other three programs. Two of these six groups received the guided discovery program together with the formula program; two additional groups received the expository and formula programs; and the final two groups received the opportunity-to-discover and formula programs. One group, in each of the resulting three pairs, received the programs in one order; the other group received them in the reverse order. Only the order of presentation was varied. After finishing their respective programs, all of the students were tested on new series to see how well they could determine the appropriate summing formulas.

The results were rather clear cut. Essentially, the group given the formula program only and the group given the formula program followed by the opportunity to discover program performed at one level. The other five groups performed at a common and significantly higher level. Two points need to be emphasized. First, "what is learned" during guided discovery learning can at least sometimes be taught by exposition--with equivalent results. Of course, there are undoubtedly a large number of situations where because of the complexity of the situation, "what is learned" during discovery can not be clearly identified. It is still an open question, for example, whether still higher order derivation rules, which have a more general effect on the ability to learn, may be learned by discovery. If we believe that the answer to this question is affirmative, there is no real alternative to learning by discovery unless or until we can identify just what is involved. Nonetheless, intuition-based claims that learning by self-discovery produces superior ability to solve new problems, as opposed to learning by exposition, has not withstood experimental test. The value of some forms of discovery to transfer ability does not appear to exceed the value of some forms of exposition. Apparently, the discovery myth has come into being not so much because teaching by exposition is a poor technique as such but because what has typically been taught by exposition leaves much to be desired. As we identify just what it is that is learned by discovery in more and more situations, we shall be in an increasingly better position to impart that same knowledge by exposition.

The second point to be emphasized concerns the sequence effect. While the group that was given an opportunity to discover and then the formula program performed as well on the transfer problems as those given the derivation principle in a more direct fashion, the group given these programs in the reverse order (i.e., the formula-opportunity group) did no better than those Ss given the formula program alone. In effect, if a person already knows the desired responses, then he is likely not to discover a more general derivation rule.

An extrapolation of this result suggests that if S knows a specific rule, then he may not learn one which is more general even if he has all of the prerequisites and is given the opportunity to do so. The reverse order of presentation may enhance discovery without making it more difficult to learn more specific rules at a later time. In effect, prior knowledge may actually interfere in a very substantial way with later opportunities to discover.[19]

This sequencing result may have important practical and theoretical implications. The practical implications will be attested to by any junior high school mathematics teacher who has attempted to teach the "meaning" underlying the various computational algorithms after the children have already learned to compute. The children, in effect, must say to themselves something like, "I already know how to get the answer. Why should I care why the procedure works?" Similarly, drilling students in their multiplication facts before they know what it means to multiply, may interfere with their later learning what multiplication is. Let me make this point clear, because it is an important one. I am *not* saying that we should teach meaning first simply out of some sort of dislike for *rote* learning--for certain purposes rote learning may be quite adequate and the most efficient procedure to follow. What I *am* saying is that learning such things as how to multiply without knowing what multiplication means, may actually make it *more* difficult to learn the underlying meaning later on. The theoretical implications are even more interesting for the researcher and, in fact, may be crucial to any theory based on the rule construct and framed in the SFL--but space limitations demand that I not go into that here.

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[19] In spite of this fact there may be some advantages inherent in learning more specific rules. Although the data are not entirely clear on this point, it is quite possible that specific rules may make it possible to determine responses more quickly than rules which are more general.

In addition to the studies described above, we have conducted a number of other studies, which are based on the SFL, but space limitations make it impossible to more than mention them here. One of these studies is designed to help clarify the role of attribute and operation cueing in learning mathematical rules. Another deals with the role symbolism plays in mathematics learning. We are also involved in developing a completely new methods course in mathematics for elementary school teachers which is intimately tied to this point of view.

Concluding Remarks

In this paper, I have tried to share with you some of my thoughts on the psychology of mathematics learning--or what I like to think of as the *emerging* discipline of *psycho-mathematics*. As is obvious by now, I chose not to do this in a direct manner but by pointing out certain inadequacies in existing behavioristic theory as it relates to mathematics learning and, more important, by describing an alternative scientific language and showing how it can be used to formulate research questions involving mathematical learning and performance.

The mathematical notions of sets and functions were proposed as a basis for representing: (1) the denotative or observable aspect of rules and principles, (2) the underlying knowledge itself, and (3) meta-linguistic descriptions or representations of the underlying knowledge. Very recently, I have become intrigued with the idea of integrating these ideas by borrowing the very fundamental but more abstruse mathematical idea of a functor. The functor may also make it possible to distinguish in a very precise way between the sort of "ideal" competencies which have long been championed by linguists and competencies as they actually exist in human beings. In the present version of the SFL, it has only been possible to deal with idealized rules (i.e., competencies).

To insure continued progress, it seems to me that a dual emphasis is needed in psycho-mathematics. On the one hand, there are a large number of unspecified, but crucial, "ideal" competencies which underlie mathematical behavior. These need to be identified. This is a problem area which in many ways is analogous to linguistics and whose solution will require a substantial knowledge of mathematics coupled with a behavioral point of view. There is also the urgent need to consider how the inherent capacities of learners and their previously acquired knowledge interact with new input to produce mathematical learning and performance. Again by analogy, we have a field much like psycho-linguistics, a field which will seek to integrate knowledge concerning psychology and mathematical structures and strategies. I feel that this kind of distinction will prove crucial to any deep understanding of how mathematics (and other subject matters) are learned.

PRECISION IN RESEARCH ON COMPLEX LEARNING AND TEACHING--
THE MATHEMATICAL FORMULATION OF EDUCATIONAL RESEARCH QUESTIONS

Joseph M. Scandura

Proj. No. OEC-1-7-168002-0165

University of Pennsylvania

July 1, 1966 to June 30, 1967

Background

In spite of an increasing number of basic research studies, theoretical development in educational psychology has been extremely slow. With few exceptions, the research on meaningful learning has been fragmentary. Results have often appeared to be inconsistent; similarities and essential differences have gone undetected.

A major reason for this slow progress has been the lack of an appropriate scientific language with which to discuss educational research problems. Existing languages in psychology fail to capture the essence of meaningful learning and teaching. The choice to date has been between a precise, but seemingly inappropriate S-R language, and presumably more relevant cognitive formulations which leave much to be desired insofar as scientific cohesiveness and rigor are concerned.

At the time this project was initiated, preliminary steps had been taken toward the development of a new scientific language for formulating research questions on meaningful learning. Because the language was framed in terms of the mathematical notions of sets and functions, the name Set-Function Language (SFL) was adopted.

Objectives

The overall objective of this research was to extend, elaborate, and otherwise improve the SFL. We sought to determine its strengths and weaknesses by applying it to a variety of complex learning situations, particularly those involving mathematics.

More specifically, we proposed to :

(1) contrast the SFL with the S-R language,

(2) present a methodology for assessing what is learned,

(3) (re) formulate a variety of research questions in terms of the SFL and the related assessment methodology, and

(4) describe some empirical research which was based on a preliminary formulation of the SFL.

Procedure

This project required a good deal of developmental activity and evolved over a long period of time. The steps taken reflect this fact.

Our preliminary thoughts on the SFL are included in paper I--"The Basic Unit In Meaningful Learning--Association Or Principle?" In this paper, we were concerned with describing the SFL and its relationship to the S-R Mediation Language. We presented a methodology for assessing what is learned and (re) formulated a variety of research questions in terms of the SFL and the related assessment methodology.

As our understanding of the problem deepened, we undertook the difficult task of refining and extending the SFL in paper II, "Using the Principle to Formulate Research on Meaningful Learning." We were able to do this by characterizing a principle as an ordered four-tuple, (I,D,O,R), where: (1) D referred to those stimulus properties which determine the responses, (2) O, to the combining operation by which (3) the response properties (R) are derived from those in D, and (4) I, to those stimulus properties which determine when the rule, $O(D) = R$, is to be applied. We also showed in this paper how the SFL aided in the development of some of our research and how problems concerning reception and discovery learning, reversal and nonreversal shifts, Piagetian conservation tasks, and symbolic and concrete learning can be reformulated in the SFL.

At about this time, we felt the need for a commentary to researchers in mathematics and science education. In paper III, "Precision in Research on Mathematics Learning: The Emerging Field of Psycho-Mathematics," we attempted to provide in-depth analyses of certain questions of concern to all mathematics and science educators. We indicated how the SFL may help the researcher in his quest for new and important questions, and how it may be of even more help in formulating his questions in researchable forms.

Paper IV, "The Basic Unit In Meaningful Learning--Association Or Principle?" contrasts SFL and S-R formulations of several meaningful learning tasks. These tasks were shown to be easily represented in SFL terms and difficult, if not impossible, to formulate in the S-R language. We showed how principles as well as associations and concepts can be represented in the SFL and indicated areas of SFL-based research either completed or underway during the past year.

The final paper, "Concept Learning In Mathematics (Research in the Emerging Discipline of Psycho-Mathematics)" attempts to glean some of the highlights of both theory and empirical research based on the SFL. We also suggest future directions for psycho-mathematics and comment on what may result in a major refinement of the SFL.

Results

During the year that this project was in operation, a great deal was accomplished. It would be impossible to summarize all that was learned. In brief, we have devised and refined a new scientific language (SFL) which provides for the description of psychologically relevant subject matter characteristics. We have presented evidence in favor of using the principle, rather than the association, as the basic behavioral unit in meaningful learning. A methodology, based on the SFL, was presented for assessing "what is learned." We also showed how the SFL and the related assessment methodology can be used to (re) formulate a variety of research questions (e.g. paired associate principle learning, reversal and nonreversal shifts, Piaget-type conservation tasks, and symbolic and concrete learning). In addition, a number of empirical research studies based on SFL analyses were described (e.g. Scandura, Woodward, and Lee (1967), Greeno and Scandura (1966), Roughead and Scandura (1967)). In the Roughead and Scandura (1967) study, for example, the SFL analysis of what is involved in discovering mathematical rules indicated that discovery learners learn "something" by which they can derive solutions to an entire class of problems. We called this "something" a derivation rule. Using the SFL as a guide, we were able to identify that derivation rule taught in a guided discovery program used in an experiment by Gagne and Brown (1961).

Conclusions and Implications

We have pointed out certain inadequacies in existing behavioristic theory as it relates to mathematics learning. An alternative scientific language (SFL) was described and it was shown how the SFL could be used to formulate research questions involving mathematical learning and performance. The SFL helps fill the gap between the highly controlled studies of the learning laboratory and the more encompassing but less well specified research of the mathematics educator.

There are many problems that need to be solved if a useful theory of mathematical learning is to be invented. The SFL in its present form is not sufficient to deal with all of them. For example, no attempt has been made to extend this approach to deal with instruction which simultaneously involves several objectives. Second, complex problem solving and proving theorems, in which there are several stages, have not been considered. Third, the question of whether the SFL can be used to deal with highly complex forms of interrelated knowledge is still open and, finally, the SFL says little or nothing about memory or learning efficiency (i.e. time to learn). However, none of these problems seems insurmountable and we are presently looking into the possibility of extending the SFL in two ways: (1) by using it as a basis for instructional planning in mathematics, and (2) by looking into the very intriguing possibility of reformulating the SFL in terms of the more fundamental, but abstruse, mathematical notion of a functor. Although it is too early to say what may develop, we are very optimistic.

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PRECISION IN RESEARCH ON COMPLEX LEARNING AND TEACHING-- THE MATHEMATICAL FORMULATION OF EDUCATIONAL RESEARCH QUESTIONS

Joseph M. Scandura

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U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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The field of learning psychology spans all the disciplines. This scholarly study, while primarily directing itself to the area of mathematics, will prove to be stimulating and thought provoking to all scientists.

Precision in Research on Mathematics Learning: The Emerging Field of Psycho-mathematics[*]

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The number of fascinating and weighty questions concerned with mathematics learning is certainly large, if not uncountable. As critically important as it is for the psychomathematician[*] to ask significant questions, however, that alone is not sufficient. Unless such questions are formulated so as to provide definitive information, the knowledge so acquired may be worse than no information at all—it may be misleading.

Since this problem is of no small concern in research on mathematics learning, I shall, in this paper, attempt the perhaps overly ambitious task of providing in-depth analyses of certain questions of concern to all mathematics (and science) educators. A major result of these analyses is a scientific set-function language (SFL) that may be of some help to the researcher in formulating his questions in researchable form.

To accomplish this, I shall describe some of my own research over the past few years, replete with blunders and insights. Consideration is given to: (*1*) the assessment of "what is learned," (*2*) the basic behavior unit in mathematics learning, (*3*) rule generality, (*4*) interpretability and symbolism, (*5*) concrete and symbolic learning, (*6*) cueing in discovery learning, (*7*) learning mathematics by exposition and discovery. Before I begin, however, let me give some perspective to what is being attempted by reviewing the role of basic research and theory generally, and the need for and nature of scientific languages.

Introduction

Background

The central aim of all basic, information-oriented research is theory development. Sound theory, in turn, provides the basis for many inventions of use to mankind. Einstein's Theory of Relativity provided much of the motivation for harnessing the atom, and cell theory is leading in the direction of cancer prevention—so a sound theory of mathematics learning (and teaching) may make it possible to construct instructional procedures, efficient beyond present-day expectation.

While there has been a good deal of action research in the classroom and carefully controlled comparisons of different instructional sequences or curricula, there has been little attempt to identify just what it is that makes one method or curriculum better or worse than another. To achieve a better understanding of how mathematics learning takes place, not only must the research be sound in design, but the right questions must be asked and properly formulated.*

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[*] The name "psycho-mathematics" ("psycho-mathematician") derives by analogy from psycholinguistics (psycho-linguist) and suggests a dual concern with human behavior and the structural characteristics of mathematics—the study of the behavioral invariants of mathematical structure.

I shall concentrate here on the identification and formulation of significant questions involving mathematics learning. Let me be clear as to just what is intended. The concern is not simply with the identification of problem areas (e.g., the need for a better understanding of how to sequence mathematics instruction, the role of symbolism in mathematics learning, how slow learners learn mathematics) as important as that might be. The mathematics educator, raising such questions, is much like the nuclear technician saying that we need to know more about the particles comprising atomic nuclei before we can successfully harness the power of the H-bomb. Neither is my task that of simply arguing for the clear and unambiguous specification of experimental variables. That is a necessary but not sufficient condition.

What I would like to do is consider how the psycho-mathematician is led to raise significant new questions, and once raised, how he goes about formulating them in researchable form. Unfortunately, I know of no complete answer to either of these questions. Furthermore, no *a priori* guarantee or procedure can ever be prescribed to insure that the scientist will not end up a blind alley. (This is probably a good thing since so many seemingly blind alleys have led to some of the biggest scientific breakthroughs.) Especially in the less developed sciences, of which psycho-mathematics would be one, most research questions are based solely on intuition. The adequacy of both their identification and specification are dependent on the perspicuity of the scientist's intuition.

Nonetheless, there are more formal means by which the scientist may be led to ask new questions. One of these is theory. Manipulations within the system, itself, may lead to new propositions ready for empirical test. The advantage of generating research questions on the basis of theory, rather than by sheer intuition, is that the experimental results tend to either substantiate or refute not just the question at hand but a whole set of interrelated propositions about the real world. Where such theory exists, a theoretical approach is certainly to be recommended, for economic as well as aesthetic reasons. There are simply too many questions that might otherwise be asked—and research is expensive of both time and money. As suggested above, however, the scientist must be constantly aware of the danger of restricting his concern to one theory with little attention being given to the real world. He might well end up with a highly elegant theory that says nothing in particular about anything important.

In any case, there has been very little theory based research on mathematics learning for the simple reason that there are few relevant theories. Hopefully for the future some highly promising work is underway.[1]

Rather than well-formulated deductive theories, however, the scientist may be led to ask new research questions by making use of scientific languages or other incomplete models. A scientific language is just what the name implies, a symbolic language with which to talk about scientific questions (e.g., to formulate research problems).

As with theory, there are certain desirable characteristics of scientific languages. A language should, if it is to be an improvement over the native language, be concise with the constituent symbols having a precise meaning. Mathematics makes an ideal language in this sense. The primary requirement, however, is that the language accurately represent the phenomena in question. Without such fidelity the language can have no real value. The ideal language is one which strips an empirical situation of non-essentials while accurately and precisely retaining that which is central—a process much akin to abstraction. While the mathematician asks, "Why restrict one's attention to euclidian spaces when any metric space will do?", the psycho-mathematician asks such questions as "Why restrict one's attention to the effects of familiarity, with the concepts of limit and absolute value, on learning the epsilon-delta definition of a limit when a large number of behaviorally similar situations exist in mathematics learning?"

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* This paper is concerned only incidentally with questions of design. Competence in this area is assumed (not always a feasible assumption as regards educational research). Once his problem has been identified, the technically competent scientist must be able to set up, conduct, and analyze his data so as to answer the question initially posed. Without such competence the scientist, whether physicist, chemist, psychologist, or psycho-mathematician, is like the mathematician who cannot prove routine theorems. It is his *sine qua non*.

Although it may provide a rigorous basis for analyzing empirical situations, however, a scientific language may have only indirect relevance to any particular set of theoretical postulates. This is at once the major advantage and the major limitation of a scientific language. It is an advantage in that no commitment is made to any one theory although usually some type of theory is implied. Thus, while the S-R *language* is used almost exclusively by contemporary experimental psychologists, the S-R theories employed vary from highly structured Hullian theory and Guthrie-Estes-type stochastic models to the use of relatively simple S-R diagrams with a minimal list of *ad hoc* assumptions. The major limitation of a scientific language is that empirical predictions cannot be formally derived without the addition of theoretical assumptions.

Inadequacy of Existing Languages

Granting the potential value of a language in promoting significant research on mathematics learning, the important question is, "What language?" As indicated above, an entire school of psychologists has found extremely useful the so-called S-R language, a language which, for the most part, has derived most of its basic concepts from animal experimentation. Is this molecular associationist language suitable for use in describing mathematics learning? Surely, I am not the first to think not.[9]*

Other frameworks within which to treat mathematics learning, teaching, and/or transfer have been advanced at a more molar level, but they have been intended as a means for organizing literature reviews and not as a basis for making the fine discriminations necessary in formulating research on mathematics learning. Henderson[2] for example, viewed mathematics teaching as an ordered triple, teacher X teaching subject or topic Y to student(s) Z. Similarly, Becker and McLeod[3] have proposed a basis for characterizing transfer situations in mathematics.

A most important outcome of recent collaborative efforts between mathematics educators and psychologists has been to focus attention on the close relationships between task and method variables. Mathematics educators and psychologists have increasingly come to realize that research on mathematics learning and teaching must, on the one hand, deal with mathematical structure, and, on the other hand, with observables (i.e., behavior). Little of scientific and practical value to education can be accomplished by talking about either structure-free (i.e., rote) materials or unobservable mental processes.

There has been a resulting recognition of the need for a precise language, couched in observables, which also provides for the description of psychologically relevant mathematical characteristics. To be useful, such a language should, at a minimum, make it possible to represent existing learning, what information is presented and how, criterion tasks, and structural relationships between them.

How does one go about devising a scientific language? Here again no one answer can be given. What I shall describe in the following sections is the chain of events which led me to the set-function language (SFL) there introduced. My approach was, in part, haphazard and, in part, based on an analysis of just what is involved in *certain* aspects of mathematics learning (I clearly have *not* attempted to deal with all that is relevant). Where appropriate and possible the questions considered are represented symbolically, thereby exposing the basic structure of the empirical situation involved. Happily, it turns out that these critically important characteristics can be formulated precisely, largely in terms of the set and function concepts of mathematics.

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* Further discussion of this and subsequent topics where indicated is available by corresponding with the author; he will welcome such communications. *Editor*.

Before proceeding let me emphasize that I do not pretend to have the final word, but only what I feel is a step in the right direction.

The SFL and Research on Mathematics Learning

What Is Learned and Response Consistency

If, as has been proposed,[4,13,14,24] the current state of the learner is critical in determining his response to a new stimulus, and, indeed, in determining his future learning, the assessment of "what is known" becomes a central task in psycho-mathematics.

To see what is involved, consider the situation depicted in Figure 1. Suppose an experimental subject (*S*) is required to learn to say the indicated words when shown the learning stimuli (e.g., to say "black," when shown the large black triangle). After the four training S-R pairs are mastered, so that the subject can reliably give the correct response to each stimulus, the question remains as to just what was learned. Did the subject learn four discrete pairs (i.e., associations), noticing no relationships between them? Or, did he learn the two *principles*, "If triangle, then color." and "If circle, then size."?[*]

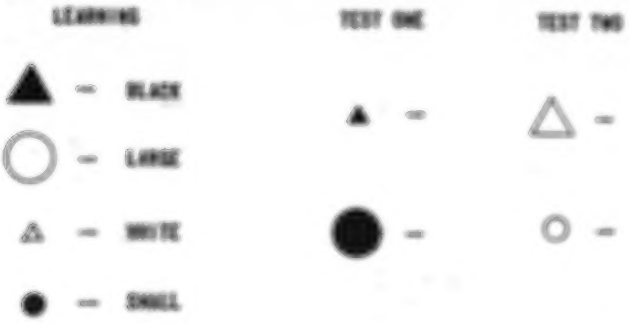

Fig 1. Sample learning, assessment (test one), and prediction (test two) stimuli and responses.

This question first began to bother me while working with Jim Greeno at Indiana University during the summer of 1962. In a study designed by Greeno,[5] we found, in a verbal concept learning situation, that essentially *S* either gives the correct response the first time he sees a transfer stimulus or the transfer item is learned in the same way as its control.

The thought later occurred to me that if transfer obtains on the first trial, if at all, then responses to additional transfer items, at least under certain conditions, should be contingent on the response given to the first transfer stimulus. In effect, a first transfer stimulus could serve as a test to determine what had been learned during stage one, thereby making it possible to predict what response *S* would give to a second transfer stimulus.

To test this assumption, I had a number of small groups of college graduates, a total of about fifteen *Ss*, overlearn a list similar to that shown in Figure 1. Prior to learning the lists, both the *Ss* and the experimenter (*E*) agreed on the relevant values and dimensions—size (large-small), color (black-white), and shape (circle-triangle). The *Ss* were told to learn the pairs as efficiently as they could since this might make it possible for them to respond appropriately to the transfer stimuli. After learning, the Test One Stimuli were presented and the *Ss* were told to respond on the basis of what they had just learned. Positive reinforcement was given no matter what the response (i.e., the *Ss* were told they were correct). The Test Two Stimuli, then, were presented in the same manner.

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[*] Notice that both statements have the form, "If I', then R'." The reason for using primes will become clear in a later section (*Further Analyses*).

The results were clear-cut. All but three of these *Ss* gave the responses "black" and "large" respectively to the two Test One Stimuli (see Fig. 1), and also responded with "white" and "small" to the Test Two Stimuli. On what basis could this happen? It was surely not a simple case of stimulus generalization;* the responses did not depend solely on common stimulus properties. The first Test One Stimulus, for example, is as much like the fourth learning stimulus as the first (see Fig. 1).

Perhaps the simplest interpretation of the obtained results is that most of the *Ss* discovered the two underlying principles while learning the original list and later applied them to the test stimuli. In effect, the common relationships between the S-R pairs when combined with a response consistency hypothesis (i.e., when *S* "thinks" he's right and the new situation is relevant, he will continue to respond in a similar manner) provided a basis for assessing "what was learned."†

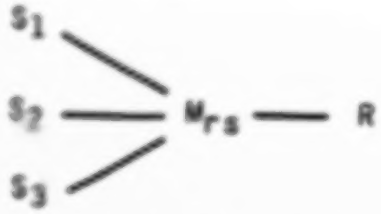
Basic Unit of Behavior—Association or Principle

In order to construct a precise descriptive language which adequately reflects mathematics learning, a basic behavior unit must be selected. The history of science has shown that the hypothesis-generating and predictive value of any theory or scientific language is determined in large part by the appropriateness of its basic building blocks.

In S-R psychology this basic building block is the *association*, a learned connection between an observable stimulus (e.g., light, nonsense syllable, or mathematical problem) and an observable consequence or response (e.g., salivation, another nonsense syllable, or solution). A connection or association is said to be formed if the response appears with a positive probability whenever the corresponding stimulus is presented.

Learning a concept, presumably a more complex form of learning, involves the ability to give a common response to any one of a set of stimuli. To say that *S* understands the concept of "circle," for example, implies that *S* is able to say the name "circle," the common response, when shown any example of a circle but will not say "circle" to any noncircle stimulus.

In effect, whereas an association is a 1-1 relationship, a concept is a many-1 relationship. Since 1-1 relationships are felt to be basic, the S-R psychologist has felt obliged to represent the *many*-1 concept relationship as a composite of 1-1 relationships. This was made possible by the postulation of mediating links (associations). Thus, the *many*-1 relationship can be viewed as



where S_1, S_2, and S_3 are stimuli connected to the mediating response M_{rs}, whose stimulus properties, in turn, elicit R.*

A still more complex form of learning is the principle. Knowing a principle makes it possible to give the appropriate response in a class of responses to any one of a class of stimuli. Principles are, thus, *many-many* relationships and operate somewhat as indicated by a statement of the form, "If A, then B."[7] Examples of principles in mathematics are easy to come by. The principle (statement), "If given a quadratic equation (of the form $x^2 + Bx + C = 0$), then the two roots, r_1 and r_2 may be obtained by solving the equation $(-B \pm \sqrt{B^2 - 4C})/2$," for example, makes it possible (assuming the necessary manipulative skills) to find the roots of any quadratic equation.

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* Stimulus generalization refers to the fact that there is a positive probability that a new stimulus, A_2, will elicit response B whenever A_2 is similar to A_1 and A_1 already is known to elicit B.

† In effect, this assessment procedure capitalizes on the *Einstellung* (mental set) phenomena, first discovered by Luchins.[8] The fact that adult *Ss* tend to encode via dimensions (e.g., color) rather than stimulus values (e.g., black), however, may be at least partially original.

As will be seen in the following study on Rule Generality, response consistency also obtains with actual mathematical rules.

* In this display, M_{rs} refers to both the mediating response and its stimulus properties. Whereas S-R psychologists have made use of this distinction, no purpose is served in doing so here.

Because of their obvious relevance to mathematics (and other meaningful) learning, it is perhaps surprising to the mathematics and science educator that psychologists have done relatively little research on principle learning. In view of the preoccupation with associations, however, it is perhaps not so surprising after all. No one has made a serious attempt to characterize principles in terms of associations.* In attempting to see if it could be done, I came up with what appears to be an accurate representation. The result,[9] however, is far from aesthetic. In fact, it is downright cumbersome—still more reason for psychologists to have almost completely avoided the study of principles.

When the observable, and therefore scientifically relevant, aspects of the association, concept, and principle are formulated mathematically, the result, in each case, is simply a set of ordered stimulus-response pairs. A learned association is denoted by only one such pair, a concept by a set of pairs with a common response, and a principle by an arbitrary *function* (i.e., a set of ordered pairs, such that each stimulus corresponds to one and only one response). When looked at in this way, *the principle becomes the basic unit while the concept and association become special cases.*

The *psycho-mathematician* is, thus, faced with a dilemma: (*1*) maintain the association as the basic unit, along with a large body of existing psychological data, and have a cumbersome tool (language) with which to study mathematics learning, (*2*) adopt the principle and thereby base his research on a relatively unproven behavioral unit, or (*3*) formulate his research questions on a strictly intuitive, and relatively imprecise, basis. Although arguments can be made for each approach, the present paper is concerned only with the second.

Principle Generality and Consistency in Mathematics Learning

How general (abstract) should the presentation of mathematics be? Should addition and subtraction be taught as two distinct, although related, operations, as has been done traditionally, or as one operation as in more modern treatments? Should the three cases of percentage be taught separately or as variants of the principle, "base times rate equals percentage?" Should pupils be taught the method of "casting out nines" or be taught the more general principles of modular arithmetic? How generally should theorems be stated? Proofs?

Mathematics educators have had to concern themselves with such problems, but they have had only intuition to guide their judgment. There is a real need to better understand the psychological principles involved. Unfortunately, however, previous studies involving rule and principle learning have dealt only incidentally with the question of generality.

Assuming that the answers hinge, at least in part, on learnability as well as general utility and armed with the denotative characterization of a principle as a function, Woodward, Lee and I[10] set out to explore this question. In particular, we were concerned with the effects of principle generality on learnability and transfer. We also explored the response consistency hypothesis with more complex materials. Two experiments were conducted, the independent variable in both cases being the scope (i.e., generality) of a *principal statement.* Scope was defined in terms of the corresponding denotation, one statement being more general than another if the denotation of the former included the latter.*

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* Gagne[7,8] has advanced a formulation of a principle in terms of concepts. While this representation is operational (i.e., useful), it is questionable whether the association is the basic building block.

Our original hypotheses were that: (*1*) the scope of a principle would be fully reflected in performance, there would be little success on extra-scope problems and no differences in performance on within-scope problems, (*2*) the learnability of a statement, as determined by within-scope performance, would vary inversely with scope, and (*3*) the combining rule taught would be used on all problems even when, after proposing a problem solution, *S* was not told whether he was right or wrong (i.e., under conditions of *nonreinforcement*).

Experiment One. In the first experiment, each group of 17 college *Ss* (majors in elementary education) was presented with one of three ordered principles dealing with a variant of the number game called NIM.[11] In the game, two players alternately select numbers from a specified set of consecutive integers, beginning with one, and keep a running sum. The winner is the one who picks the last number in a series with a predetermined sum. If, for example, this sum is 31 and the set consists of the integers 1–6 the players alternatively select numbers from 1–6 until the cumulative sum is either 31 or above (in which case no one wins). There are rules which allow the player who goes first to win.

Any such game can be characterized by an ordered pair of integers. The application of each winning principle was illustrated with a common (6, 31) game. The least general principle (S), adequate for winning only (6, 31) games, was stated, "…make 3 your first selection. Then … make selections so that the sums corresponding to your selections differ by 7." Principle (SG) was adequate for solving $(6, j)$ games $j = 1, 2, \ldots, n$ and was stated, "the first selection is determined by dividing the desired sum by 7 and making the remainder your first selection. …Then … make selections so that the sums corresponding to your selections differ by 7." The most general principle (G) was adequate for solving (i, j) games $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ and was stated, "the first selection is determined by dividing the desired sum by one more than the largest integer in the set from which the selections must come and making the remainder your first selection.… Then … make selections so that the sums corresponding to your selections differ by one more than the largest integer in the set."

All *Ss*, including two control groups, were tested on three problems. The first was within the scope of each principle, the second within the scope of all but principle S, and the third only within the scope of principle G.

The *results* were straightforward.* Of those 13 *Ss* in group S who solved problem one, none solved problem two, and only one solved problem three. The corresponding numbers for groups SG and G were, respectively, 5, 4, 0 and 5, 5, 4. Within the scope of each principle there were only chance differences in performance on the problems. On the other hand, only one *S* solved an extra-scope problem.

The relative interpretability of the three rule statements was determined by comparing group performance on problem one which was within the scope of each. Rule S proved to be easier to learn, under the self-paced conditions, than were the rules SG and G ($p < 0.007$ in both cases). There was, however, no discernable difference in the interpretability of rules SG and G.

The third facet of this research was concerned with the consistency with which presented principles are applied. We wanted to determine whether the S and SG *Ss* would use the rule taught even when it was inappropriate (on the second and third problems). To make this possible, no information was given as to when the principles were and were not appropriate.

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* Notice that defining a principle as a function makes it possible to consider a variety of other relationships between different principles. In particular, two functions (i.e., sets) may be disjoint (i.e., have no instances in common), overlap, or be identical in addition to being ordered (i.e., one being more general than the other). Such relationships might provide a basis for formalizing questions concerning the effects of prior learning on later learning.

* Exact probability tests on 2 × 2 contingency tables were used to test the various hypotheses.

Of the 17 S *Ss*, 13, 9, and 8 used the rule taught on problems one, two, and three, respectively. The corresponding numbers in groups SG and G were 7, 7, and 5 and 6, 6, and 6. Although there was a slight tendency to not use the rules taught on problems two and/or three, where they were inappropriate, there were no significant differences in frequency of use.

These results certainly provided strong support for our original hypotheses: (*1*) performance on within-scope problems did not differ appreciably, even though the common illustration was more similar to problem one than the others, and successful problem solving was limited almost exclusively to within-scope problems, (*2*) rule S proved easier to interpret than rules SG and G, and (*3*) the rules taught tended to be used consistently on all problems whether they were appropriate or not.[*]

About the only major unanticipated result in experiment one was that rule G proved as easy to interpret as rule SG. In view of the rather low proportion of successes in these groups, we were originally tempted to attribute the lack of such an effect to scale insensitivity near its lower extreme.

Experiment Two. To determine the generality of these findings, a second experiment, dealing with arithmetic series, was conducted with junior high school *Ss*. In this experiment, both scope (S, SG, G) and example (present, absent) were varied independently. Since most of the *Ss* were already familiar with the arithmetic operations introduced and, to some extent, with number series generally (i.e., as in adding lists), it was felt that examples might provide a basis for generalization, via discovery, to extra-scope problems. Another difference between this experiment and the first was that rule S, 50 × 50 (= 2500), was effectively an answer given treatment and applied to only one series. This series was used both as the common example and as problem one. In experiment one, rule S applied to a number of different game sequences.

Although the pattern of results shown in Table I paralleled those of experiment one in most respects, there were several important differences. *First*, the presence of the example (problem one) along with rule S resulted in significantly better performance on problem two than when rule S was shown alone, the only case in either experiment where nonnegligible success was noted on an extra-scope problem. This effect may have been due to the form of the combining operation, "50 × 50," in the rule S statement. "50 × 50" is clearly an instance of the more general SG combining rule, "$n \times n = n^2$." Presumably, the statement of rule S, together with the common illustrative series, $1 + 3 + 5 + \ldots + 97 + 99$, provided the successful S *Ss* with enough cues to generalize. In particular, they may have discovered that this series had 50 terms. Hindsight suggests that this difficulty might have been overcome by simply stating the sum, 2500, of the illustrative series rather than "50 × 50."

Second, Table I indicates that only three of the 19 G-with-example *Ss* solved problem three whereas 18 solved problem one and

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[*] The first mentioned result has particular relevance for the psychologist since it crystallizes the fact that no generalization gradient is to be expected when the stimulus values are discrete rather than based on a continuous physical dimension. If there were such a gradient, performance on the first test problem, which was more similar to the example, should have been superior to that on the other problems. Even S-R associationists are generally agreed that the lack of such an effect provides indirect support for a rule or principle interpretation. To the extent that the variables involved in meaningful learning are discrete, a rule interpretation may prove more useful.

Furthermore, when the underlying stimulus dimension(s) are continuous, S-R theorists will need to consider the possibility that generalization gradients are simply artifacts of averaging individual differences in perceptual discrimination (and, hence, what rule is learned) over continuous dimensions.

TABLE I

Summary: Number of Correct Sums (Uses of Rule Taught) on Problems One, Two, and Three

	Rule				Rule and Example			
	N	one	two	three	*N*	one	two	three
Group S	20	8 (8)	1 (1)	2 (1)	21	20 (20)	9 (0)	3 (0)
Group SG	20	5 (5)	4 (5)	0 (5)	15	11 (12)	8 (9)	1 (6)
Group G	19	3 (5)	5 (7)	2 (7)	19	18 (18)	14 (16)	3 (15)

14 solved problem two. The decrement between problems two and three was significant ($p < 0.004$). The reason for this difference was not immediately apparent especially since 15 of these *Ss* applied rule G to the third problem. A more intensive *post hoc* analysis of the situation, however, suggested that the result may have been due to a difference in ease of determining N, the number of terms, for use in the G combining rule, $[(A + L)/2]N$. N could be determined from problem series one and two by taking the average of the first and last terms. A careful examination of the test papers suggested that this led to the incorrect value (25, rather than 24) for N in the third series, $2 + 4 + 6 + \ldots + 46 + 48$. In short, the difficulty was not in the rule but in finding the correct value of N. Such difficulties may be circumvented in future experimentation by controlling for such unwanted differences.*

Third, although the results of experiment two were in the hypothesized direction, only the overall effect of scope on interpretability was significant. This led us to wonder whether interpretability of the principle statements depended *solely* on generality. Could the principle statements have also differed as to the difficulty of interpreting the actual terms or symbols used? After consideration of this possibility, interpretability was rejected as an important factor in experiment two since a recheck convinced us that we had succeeded reasonably well in stating each principle as clearly as possible. Perhaps a more likely interpretation is that the *Ss'* familiarity with arithmetic interacted with the materials used so as to reduce the effects of statement generality.

Fourth, only one of the *Ss* who was shown the rule, 50 × 50, applied it to problems two and three. This result can probably be attributed to an interfering effect due to prior familiarity with addition problems. The *Ss* may simply have mistrusted rule S. How could a rule like 50 × 50, having only one answer, be the sum of all three problem series? Most junior high school *Ss* would find it unreasonable that the series $1 + 3 + \ldots + 99$ (problem one) and $1 + 3 + \ldots + 79$ (problem two) have the same sum (50 × 50). Some such reluctance may also have obtained on problem one with group S-without-example. Nonetheless, we were surprised that only 8 of those 20 *Ss*, not presented with the illustrative series, gave the correct sum (2500 or 50 × 50) for problem one.

Implications and Theoretical Comment. The results of these experiments demonstrate, in a rather conclusive fashion, the behavioral relevance of principle generality. For the most part, successful performance was noted only on tasks within the scope of verbally stated principles. When principles are presented in an expository fashion, it is normally too much to expect generalization to problems to which the principle does immediately apply.†

Of perhaps even greater practical significance were the lack (there was one exception) of performance differences on within scope problems and the consistency results. The former result demonstrates that (almost) any stimulus within the scope of a principle is equally as difficult to respond to correctly as any other. Furthermore, coupled with the consistency data cited above, the obtained consistency results suggest that *only one* (new) test stimulus is needed to determine whether, in fact, a given principle has been learned. No more information is gained by using additional test instances. *These results could have far-reaching implications for the development of highly efficient measuring instruments.*

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* It may be desirable to think of properties, such as N, as being derived from lower order (i.e., more easily discernible) stimulus properties. Thus, the rule, $(A + L)/2$, worked for problems one and two whereas $L/2$ was required for problem three.

† We have an experiment underway which we hope will help to identify the conditions under which extra-scope generalization may be expected.

In addition, the pronounced tendency of the *Ss* to attack all of the test problems in the same way, irrespective of whether the procedure used was appropriate, suggests that the ability (*i.e.*, knowing how) to solve problems and knowing *when* to make use of this ability to solve problems are quite distinct. Testing for the latter ability necessarily must involve the presentation of extra-scope problems.

Perhaps even more important than the results of these exploratory experiments were the *post hoc* analyses they made both necessary and possible. In particular, the preceding discussion strongly suggests that the roles played by various aspects of a principle statement need to be more clearly specified. The form, "If I′, then R′" does not detail all that appears relevant. For one thing, it was not possible in the rule generality study to distinguish between the roles played by A, L, and N (where A, L, and N have a particular meaning) and the algebraic expression, $[(X + Y)/2]Z$ (where the variables have general relevance). The former variables relate to properties of the series stimuli, while the latter is a ternary operation by which another such property (e.g., sums) may be derived. I′, of course, although it played no role in the rule generality study is also critical. It tells when, in fact, a rule can and cannot be applied. Thus, the rule, N^2, is appropriate whenever an arithmetic series consists of the consecutive odd integers beginning with 1 while $[(A + L)/2]N$ works whenever there is a common difference between adjacent terms.

an arithmetic series consists of the consecutive odd integers beginning with 1 while $[(A + L)/2]N$ works whenever there is a common difference between adjacent terms.

These observations suggest that a principle statement may be represented more appropriately by the form, "If I′, then O′ (D′) = R′," where I′ refers to the set of stimulus properties which indicate when the rule, denoted O′ (D′), should be applied, D′ refers to the set of those properties which determine the responses, and O′, the operation from which the responses, denoted by R′, may be derived from the properties referred to by D′.

Similarly, a principle, that internalized representation which determines a learner's responses to stimuli, may be characterized by an ordered four-tuple (I, D, O, R). Primes may be used (as has already been done implicitly) to distinguish between referents (e.g., O) and the symbols used to represent them (e.g., O′, O″).

Although the actual symbols used in a statement may be an important factor as has been suggested above and as will be demonstrated in the next section, the hypothesis advanced in the rule generality study to the effect that scope and learnability are inversely related finds a formal rationale in the nature of the characterizing elements. Making operational use, for example, of the arithmetic series property (*i.e.*, dimension), "the difference between adjacent terms is some common value," necessarily presumes that, "the difference between adjacent terms is *two*," "... *three*," "etc.," can all be correctly interpreted. The converse does not necessarily follow. A similar relationship exists with respect to the rules, 50×50 and $N \times N$. To correctly apply the latter, more general, rule to any particular series requires the ability to determine *any* value of the dimension N, including 50. Being able to apply 50×50, however, does not.

It would appear that the more general

the principle the more is expected of the learner. Whether such differences will be reflected in behavior, however, may depend on not only rule generality but the population involved, particularly on whether the *Ss* have the necessary requisite abilities.

In effect, differences in generality appear, on analysis, to be equivalent to differences in abstraction *level*. Thus, the number *two* is more abstract than the property *two oranges* because the former applies to a collection of sets only one (subcollection) of which has the latter property. For the same reason, the property represented by the place holder X is more abstract than the number *two* since it refers to a still higher order collection. Unfortunately, we have not yet conducted a study designed to provide definitive information on these points. For the present, this analysis remains hypothetical.

Interpretability and Symbolism

To help clarify the role symbolism plays in mathematics learning, I recently completed a study[13] with the help of John Davis in which we varied both the symbols actually used to construct statements of a principle and the ability of an *S* to interpret these symbols. It seems almost axiomatic that the ability to interpret a statement of principle depends critically on the ability to interpret the symbols of which it is composed,[14] be they mathematical symbols or elements of the native language (e.g., English). Nonetheless, in mathematics learning the use of mathematical symbolism is frequently, if not always, preferred to ordinary English. The reason why, however, is never made explicit.

The purpose of this study was to determine whether: (*1*) principles are more easily memorized when stated symbolically or when stated verbally* and (*2*) the ability to correctly use constituent symbols and the required (grammatical) combining rules is a necessary and/or sufficient condition for applying a learned (*i.e.*, memorized) principle statement.

Method. Four artificial principles, each unfamiliar to the 24 *Ss* (college majors in elementary education), were selected for study. Each principle was based on one of the following notions: greatest integer, sigma notation for sequential addition, vector, and partial derivative. Two statements of each principle were prepared; one was composed of unfamiliar mathematical symbolism and the other of carefully, yet succinctly, worded English. For example, the greatest integer rule was stated (in English),

1. Take the greatest integer in X.

2. Take the greatest integer in Y.

3. Divide the result of step one by the result of step two.

4. Take the greatest integer in the quotient obtained in step three.

The symbolic form of this rule was $[([X] \div [Y])]$.

Tasks also were designed to train the *Ss* to interpret the constituent symbols. For example, one set of tasks involved the greatest integer function (i.e., $\{(W, [W]) \mid \text{all real } W\}$). In addition, all of the *Ss* were required to demonstrate proficiency in the use of parentheses as a means of signifying the order in which binary operations are to be taken. These conventional rules of grammar were involved in all four principles. Of course, *neutral* materials were used to teach and assess proficiency in the use of parentheses. In no case did the pretraining or assessment include either a complete rule or one of its instances.

A 2 × 2 factorial design, with repeated measures, was used. Each *S* effectively served as his own control. One factor was the form in which a given principle was stated, symbolic or English. The other factor involved the presence or absence of training on the constituent symbols. Of course, the principles were counterbalanced over treatments so that each was used equally often under each of the four treatments. All other unwanted factors were randomized, including presentation order.

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* See footnote page 255. *Editor.*

Separate measures of learning rate and interpretability were obtained. Learning rate was determined by presenting each principle for a fixed period of time for study and testing to see if the *Ss* could completely reproduce them in written form. All four principle statements were shown once before the next go-through (trial) began. Ease of learning was determined by the number of trials it took to learn each principle to a criterion of two perfect reproductions in a row.

Interpretability was measured immediately after all of the statements had been well-learned. To demonstrate his "understanding" of the statements, *S* was required to apply each of the corresponding (underlying) principles to two stimulus instances. For example, one of the problems used to determine whether *S* could apply the integer rule was stated simply, "If $x = 8.64$ and $y = 3.24$ then . . ." All four principles were tested once before the second set of test problems was given.

Results. The results demonstrated quite clearly that: (*1*) symbolic rules are learned more rapidly, whether the constituent symbols are familiar or not ($p < 0.01$)—there were only 2 exceptions (out of 24) to this generalization, (*2*) rules, stated in symbolic form, are applied successfully if and only if the *Ss* have been taught how to apply the constituent symbols (and the necessary grammatical rules)—there were only 4 exceptions to the sufficiency part of this generalization and none as regards necessity, (*3*) rules, stated in the native English language, are applied equally well whether or not training in the use of the corresponding mathematical symbols is given, and (*4*) English statements, once learned, are applied equally as well (in this study, somewhat better) as symbolic statements in which use of the constituent symbols has previously been mastered.

These results are not entirely surprising but they do, nonetheless, make explicit at least one aspect of the role symbolism plays in mathematics learning. The use of symbolism makes mathematics learning more efficient when the constituent symbols and grammatical combining rules have previously been mastered. Symbols, of course, also serve the practical function of requiring less space in printing. Nonetheless, these results suggest that under certain conditions it may be well to remember that ordinary English can be used to teach mathematical ideas.*

Concrete and Symbolic Learning

In the preceding section, the to-be-learned principles were stated either in mathematical symbolism or in the English language. Further, the principles taught were applied to *symbolic* stimuli, symbols which were stimulus instances of one of the arbitrary principles. It was determined that a learned principle statement could be applied only when the constituent symbols were interpretable in terms of their less abstract symbolic referents. Thus, the ability to use the greatest integer rule depended on *S*'s being able to compute, say $[8.64]$, having been given the meaning of $[X]$ (i.e., take the greatest integer in X). *In short, we have so far been concerned exclusively with symbolic representations; no concrete, or even iconic, forms have been considered.*

Many stimuli, however, are symbolic representations of an abstraction with concrete referents. A stimulus such as

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* These results provide a rational basis for making one type of branching decision that, while intuitively obvious, needs to be made explicit in computer-assisted instruction. Given the objective of learning a particular principle and an expository mode of instruction, one might proceed as follows: (*1*) test to see if *S* can make use of the constituent symbols; (*2*) if so, present the principle in the more efficient symbolic form; (*3*) if not, present the principle in English.

Although learner feedback has long been recognized as an important factor in promoting efficient learning, it has been unclear as what sort of feedback to measure. *The present results suggest that specific sorts of feedback are needed in order to make specific kinds of decisions.*

"1 + 3 + 5 + 7," for example, symbolizes an abstraction reflecting the structure of a variety of more *concrete* stimulus situations —e.g., four stacks of pennies, the first containing one penny, the second three, the third five, and the fourth seven; a figure representing the produce of four countries, . . Thus, a learned principle statement is at least potentially applicable to *concrete* stimulus referents.

Suppose a young child has been taught a principle which makes it possible to say "16" when shown "1 + 3 + 5 + 7." What happens when he is presented with the four stacks of pennies and is asked how many there are? The answer to this question undoubtedly depends largely on the significance to *S* of the number symbols (*i.e.*, numerals) in the symbolic stimulus. If the numerals refer to properties of collections of sets, each including a common number of elements or objects, and "+" signifies combining, positive transfer would not be unexpected. If, on the other hand, the numerals and the arithmetic operation of addition has been learned entirely without concrete referents, say with flash cards, one could feel fairly certain that *S* would see no such relationship.

Fortunately, this question can be formulated precisely by characterizing the principles involved in the set-function language (SFL). Assume that the principle, corresponding to the symbolic stimulus, has been determined, by assessment procedures, to be (I = {1, 3, 5, 7}, D = {1, 3, 5, 7}, R = {16}, O = sequential addition). The requisite for applying this (symbolic) principle, once learned, in a *concrete* situation is precisely that principle which makes it possible to go from the concrete situation to the corresponding numbers. A *composite principle*, including this principle, along with that corresponding to the symbolic stimulus above, might be characterized (I = {1 penny, . . . , 7 pennies}, D = {. . .}, R = {16 pennies}, O = translate, for all x, the property X pennies into the higher order (more abstract) property x (the number), perform repeated addition, and translate x back into X pennies). Learning the symbolic principle statement, without being able to recognize its concrete referents, would be like having an egg shell but no egg.

The relatively simple hierarchical SFL analysis proposed provides, I feel, but a prelude to the insights which may eventuate from similar analyses in other situations. Even partial clarification of the relative roles of symbolism and concrete referents in mathematics learning and performance could have important practical as well as theoretical implications and is long overdue.*

Attribute and Operation Cueing in the Discovery of Mathematical Rules

So far we have limited our illustrations and discussion to reception learning; that is, learning which takes place via the interpretation of symbolic statements.† A good deal of mathematics, however, particularly in more modern treatments, is taught by discovery. The typical procedure involves presenting, one at a time, either stimuli or stimulus-response pairs corresponding to the to-be-discovered principle. To determine whether learning has taken place, the learner is usually asked to give the appropriate response to new stimulus instances. The former type of situation, in which *S* is tested repeatedly, is exemplified by,

The sum of the first *2* odd integers, 1 + 3 = ?

The sum of the first *3* odd integers, 1 + 3 + 5 = ?

. . .

. . .

. . .

The learner, of course, would be expected to discover that the correct sum may be obtained by simply squaring the number of terms in the (odd integer) series.

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* This sort of analysis may also prove useful in science.

† As indicated in the previous section, other variants of reception learning are possible, such as learning from diagrams or pictures (icons), as in geometry, and from concrete objects, such as with Dienes' multiple embodiments.[illegible]

Ostensibly to speed the discovery process, the teacher or auto-instructor might cue the critical aspects of the stimuli. Thus, underlining the number of odd integers (as above) or printing them in red would presumably attract attention by making the cues more salient. Of course, it would be equally possible to identify the appropriate operation (i.e., squaring) or, to introduce various combinations of both determining (D) and operation (O) cues.

Although a good deal of verbal and nonverbal cuing goes on during the contemporary discovery lesson in mathematics, there have been very few attempts to uncover the basic mechanisms involved. Whereas, recently, there have been a few such studies concerned with concept learning[illegible] and, earlier, with problem solving[illegible] no consideration has been given to principles. This is indeed, unfortunate since principles seem to underlie so much of mathematics learning.

With this in mind, the members of my research seminar on mathematics learning* and I conducted a pilot study to determine the effects of verbal attribute and operation cuing on the rate of discovering mathematical principles. In particular, the study extended that of Haygood and Bourne[illegible] in two ways: (*1*) principles were used instead of concepts and (*2*) the operations involved were arithmetic, rather than logical. The study was designed simply to determine whether identifying the determining attributes (*i.e.*, D, the stimulus attributes which determine the responses) or the appropriate combining operation (*i.e.*, O) does, in fact, increase the rate at which arithmetical principles are discovered.†

Method. To minimize the effects of individual differences, we again used artificial materials. The stimuli were four-tuples of numbers [e.g., (4, 8, 9, 3)] and the responses were simply new integers that could be derived uniquely from exactly three of the four original integers by some combination of two (of the four) elementary arithmetic operations.

Three principles were used. The determining characteristics and operations, respectively, were (*1*) A[illegible], A[illegible], A[illegible]; (*2*) A[illegible], A[illegible], A[illegible]; (*3*) A[illegible], A[illegible], A[illegible]; and (*1*) $X + Y - Z$, (*2*) $X \cdot Y + Z$, (*3*) $X \cdot Y + Z$ where the subscripts, $i = 1, 2, 3, 4$, in A_i refer to position in the four-tuple and X, Y, and Z to place holders.

A 3 × 3 design, with repeated measures on the second factor, was used. Factor one involved the type of cue given [none, determining attribute (D), operation (O)]. The second factor was a composite of the principle in question and the order of presentation* and gave some indication of the effects of one discovery on the next. The 36 elementary education majors were randomly assigned to one of the three cue groups so that a given *S* was exposed to only one type of cue and each *S* completed three discovery episodes.

The *Ss* were told that their job was to write that number which they thought corresponded to the four-tuple shown. They were also instructed, "There is a procedure by which you can always determine the corresponding number when I show you the set." Then, the control group was told, "To help you discover this procedure as rapidly as possible, you should try to determine the three specific positions in the four-tuple and a rule which combines the numbers in these positions to yield the corresponding number." The attribute group was told, "... determine a rule which combines the numbers in the (proper positions inserted) to yield the corresponding number." The rule group was told, "... determine the three specific positions in the four-tuple from which the numbers *X*, *Y*, and *Z* are always taken where (proper rule inserted) yields the correct number."

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* Mike Bendrick, John Davis, Rosalie Jensen, Bob King, Frank Pavlick, and Larry Smith.

† Although the order in which stimuli are introduced is also used extensively to promote discovery, this factor was not considered in this preliminary effort.

* No attempt was made to remove this confounding, by counterbalancing principles over order of presentation, since the study was designed as much as a learning experience (for my graduate students) as one of advancing knowledge. To have gotten too deeply involved in the details of design so early in their graduate research training could well have been self-defeating. I wanted them to think of research, first and foremost, as a conceptual experience rather than as a composite of technologies.

After *S* responded to a given four-tuple, either by writing a number or by indicating he "didn't know," the card on which the four-tuple appeared was turned over exposing the same four-tuple together with the correct number response. The *Ss* were given approximately 12 seconds to compare the four-tuple with its solution before the next four-tuple was shown. Ten such four-tuples comprised one problem and *S*'s score was the total number of correct responses made.

The probability of giving a correct response, by chance, without discovering the corresponding principle was relatively small. This was evidenced by the fact that once a correct response was given, *S* almost invariably gave the correct response thereafter.

Results. The results are summarized in Table II.

Both attribute and operation cueing induced significantly ($p < 0.01$) earlier discovery of all three principles. But, whereas a significant improvement, presumably due to practice, was noted between problems two and three ($p < 0.01$), there was essentially no difference in performance on the first two problems. This is surprising since so-called "warm-up" effects typically have their greatest effect in the beginning.

More intensive comparisons, however, indicated that problem one practice significantly ($p < 0.01$) improved operation group performance on problem two but actually hindered (significantly, $p < 0.01$) attribute group performance—classic cases of positive and negative transfer. At least two possible interpretations may be given for the latter finding. First, having discovered that some combination of addition and subtraction ($A_1 + A_3 - A_4$) worked on problem one, many of the attribute-cue *Ss* may have spent too much time trying various combinations of addition and subtraction on problem two. Second, the rule, $A_1 \cdot A_2 + A_4$, needed to solve problem two, since it involved multiplication and division as opposed to addition and subtraction, may have been intrinsically harder than that needed on problem one. Of course, both interpretations may have some degree of truth. Having been given the appropriate rules, in both cases, the rule-cue *Ss* were not subject to such effects. Furthermore, whatever response set developed on the basis of discovering that positions 1, 3, and 4 were relevant on problem one was more than compensated for by the practice afforded.

TABLE II
Mean Number of Correct Responses

	Problem		
	One	Two	Three
Attribute	6.6	4.2	8.1
Operation	3.5	7.2	8.7
Control	1.1	0.6	4.0

Clearly, research, aimed specifically at such questions, is needed to provide definitive information. It is impossible to say, at this time, that we fully understand exactly what is involved or, even more important, what the boundary conditions for these findings are.

Since specifying boundary conditions has all too frequently been passed over or, at most, been paid ambiguous lip service in educational research, I should like to emphasize one point about these results. It is doubtful that attribute cueing will ever be shown to be unconditionally better than operation cueing or vice versa. What future research may be expected to do, however, is to specify the conditions under which each will be superior.

An Analysis of Learning Mathematics by Exposition and Discovery

With the machinery and body of data built up, we are now in a position to provide a fairly detailed analysis of learning by exposition and discovery. Because of space limitations, I shall limit my analysis to the question, "*Is it better to learn by exposition or by discovery and, if there is no unique answer to this question, what are some of the conditions under which a particular method will be better?*" *In this regard, only the contention that learning by discovery enhances the learner's ability to solve new problems will be considered.* Before attempting to answer this question, let me reemphasize the differences between reception learning and learning by discovery.

Reception learning involves the encoding of information presented directly to the learner. Usually, but not necessarily, such information is presented in statement form. Understanding is said to occur, if having learned a statement so that it can be "parroted back" on cue, *S* is also able to use it to successfully determine the responses to the specified set of stimuli. As formulated in the SFL, *S* is presented with a statement that *may* be put in the form, "If I′, then O′(D′) = R′." Then, having memorized the statement, *S* is tested on stimuli arbitrarily selected from the domain (*i.e.*, the set of stimuli) of the corresponding denotative principle, $\{(S_i, R_i) | i \epsilon I\}$.

Learning by discovery, on the other hand, requires that the learner abstract the common principle from a *subset* of the S-R pairs in the set $\{(S_i, R_i) | i \epsilon I\}$. He must, therefore, identify the determining characteristics, D, the operation, O, and, if required to *discriminate* between exemplar and nonexemplar stimuli, he must also determine the identifying characteristics, I. In effect, while reception learning presupposes the ability to interpret and apply that information represented by the symbols I′, D′, O′, R′, learning by discovery requires the ability to identify, in a *nonverbal fashion*, and to use I, D, O, and R.

Unfortunately, most existing studies comparing exposition and discovery have little to say about whether discovery enhances the ability to solve new problems as is so often proposed by pedagogical enthusiasts.[4, 9] Suppose, as is often done, the exposition group is presented directly with a rule or principle whereas the discovery group is simply presented with stimuli, to which the principle relates, and required to determine the responses.* Presumably, this is accomplished by discovering the associated principle. The two groups are then frequently tested with the original stimuli (*e.g.*, problems), new stimuli within the domain of the principles involved, and stimuli requiring new principles for their solution. What is gained by comparing the expository and discovery groups on several measures, however, is lost in another way. Whenever differences in performance on within-scope problems are found, comparisons on any more general transfer tests will necessarily be *inconclusive*. Thus, obtained differences in the ability to obtain solutions to new problems, based on new principles, might equally well be due to differences in original learning as due to any advantages inherent in a method itself. In short, such studies can never hope to determine whether discovery methods actually improve the learner's capacity for dealing with new problems.

To make a valid comparison of the sort described, exposition and discovery groups must be equated on original learning. Suppose experimental *S*s are set with the task of learning several principles. In this case, the appropriate tasks on which to equate exposition and discovery groups are those which require the ability to apply the principles taught (or discovered). Assuming the necessary equivalence, tasks, requiring new principles for their solution, may be posed to compare the groups as to the ability to deal with new problems (based on new principles).

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* It is not uncommon in such studies, in fact, rather typical, for the rule-given group to perform better on within-scope problems. The symbolism study, described above, however, demonstrates quite clearly that this need not necessarily be so. If the constituent symbols are not familiar to the learner, his performance on within-, as well as extra-, scope test problems may be expected to be uniformly poor.

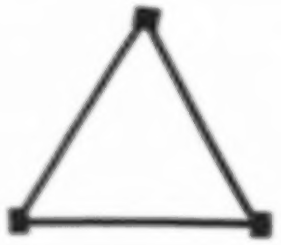
In order to predict how exposition and discovery groups would fare on such tasks, explicit consideration must be given to what is required of the learner in the two situations and the nature of the transfer problems. Discovery *Ss* must learn how to *derive* principles (those involved in the learning situation) in order to achieve criterion; exposition *Ss* may not. Making predictions, then, really boils down to *what discovery Ss learn that exposition Ss might not.* It is likely that discovery *Ss*, in attaining criterion, may discover a *derivation principle* by which new principles, similar to those originally learned, can also be derived. In this case, discovery *Ss* might be expected to perform better than expository *Ss* on tasks which can be solved via principles within the scope of the *derivation principle* discovered. On the other hand, discovery *Ss* would probably have no special advantage on problems, requiring for their solution, principles beyond the scope of this *derivation principle*.

Suppose, for example, that exposition and discovery groups are set with the task of determining the sums of number series —long series so that it is infeasible to perform sequential addition. Further suppose that two formulas (i.e., rules), say n^2 and $n^2 + n$, will suffice for all of the learning series. The discovery *Ss*, of course, would have to discover these formulas whereas they would simply be presented to the exposition *Ss*. Assuming equal mastery by the two groups, one would make the following predictions. If the formula for obtaining the sum of a new transfer series may be derived in the same manner (i.e., via the same derivation principle) as the two original formulas, then the discovery *Ss* may be expected to demonstrate superior ability. If not, no such difference may be expected.

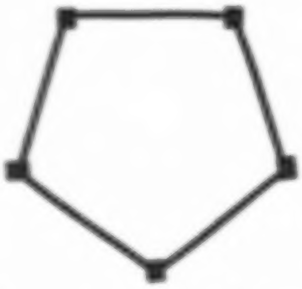
Consider a second example in which the *Ss* are to demonstrate their ability to, say, construct one of those models corresponding to each of two finite geometries.* Suppose the corresponding axiom systems are:

A. 1 there exists at least one point.
   2 each point is on exactly two lines.
   3 each line contains exactly two points.
   4 two points determine at most one line.
   5 there are no two lines not having a point in common.
B. 1, 2, 3, and 4 identical to system A.
   5 to each line, there corresponds exactly two lines which do not have a point in common with it.

then, the models

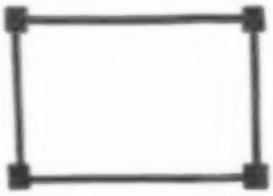


and



correspond to systems A and B, respectively.

The task posed might be viewed as involving two discrete principles, each having a denotation consisting of one stimulus (i.e., list of axioms) and one response (i.e., model). If the exposition *Ss* are presented with the models directly while the discovery *Ss* are required to derive them, it would not be surprising if the discovery *Ss* could derive the model



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* It is assumed, of course, that referents, such as "·", "–", and "—", have been assigned to the undefined terms.

for the system,

C. 1, 2, 3, and 4 identical to systems A and B.
   5 to each line, there corresponds exactly one line which does not have a point in common with it,

while the exposition *S*s could not. On the other hand, an axiom system of the form,

D. 1 there exists exactly nine points,
   2 each line contains at least three points,
   3 . . .,

might well pose equivalent difficulties for both groups.

It appears, on analysis, that these experiments were rigged in favor of discovery. The discovery *S*s were required to learn a principle(s) for deriving solutions; the exposition *S*s were not. Of course, nature is not always as simple as we would like so that it is, indeed, highly encouraging that a well-controlled study by Gagné and Brown[21] provides strong support for the essential correctness of the analysis proposed. Learning by discovery does, in fact, seem to improve ability to solve new (presumably within-scope) problems when original learning is controlled.

Nonetheless, consider what happens when we bring guidance into the discovery situation. Hints might be given, for example, to cue the determining attributes, D. They might also be used to direct the learner towards the appropriate combining operation, O; or, towards I. In fact, the *S*s might be given all of this information directly. But, then, is not this something very much akin to teaching by exposition? It would appear to be at least theoretically possible to present *derivation principles* in an expository manner rather than to depend on discovery.

So far, I have avoided the $64 question—just what are derivation principles and can they be stated in expository form? It turns out that derivation rules can be specified for the illustrations cited. The rule for deriving formulas for obtaining the sum of any arithmetic series can be stated, "Write above and below each term of the series, respectively, its position number (n) and the cumulative sum of the series through that term of the series ($\overset{n}{\Sigma}$), consider the sequence of ratios, $\overset{n}{\Sigma}/n$, and express (via induction) the general term of this sequence, as a function, $f(n)$, of n, then $\overset{n}{\Sigma}/n = f(n)$ so $\overset{n}{\Sigma} = n \cdot f(n)$ from which the sum of the (presumably too long to add) series may be determined by substitution."*

A still simpler derivation principle would work in the case of the finite geometries example. Let the determining attribute be the *number* of lines *not* having a point in common with a given line (axiom 5). The combining operation, O, would be that mapping which takes this number into a regular polygon with *three more sides* than the number in question. Whether this specification concurs with what the reader feels that a mathematics student *should* learn in such a situation is not the point. I, too, would be unhappy if this is all he learned. The point is that he *could* solve the problems posed, armed only with the simple derivation rule indicated.†

As suggested by the results of the symbolism study, being able to specify and state a higher order derivation rule in verbal (i.e., symbolic) form may not always be sufficient to insure learning. Among other things, the language used must be interpretable by the learner. *S* might not, for example, know to what the symbol "Σ" refers. Unless the derivation rule can be specified in a symbolic form that the learner can correctly interpret, there is no alternative to learning by discovery. If, on the other hand, a comprehensible symbolic (or iconic) representation can be given, the question of whether exposition or discovery is to be preferred will depend on factors, other than immediate performance capabilities, such as learning efficiency and retention—two questions to which we have not addressed ourselves.

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* This derivation procedure closely parallels that used in Gagné and Brown's discovery treatments.[21] The author is indebted to Robert M. Gagné for making his original materials available for analysis.

Such formulas, or better still the sums, might also be derived directly from the formula $[(A + L)/2]N$, where A refers to the first term of the series and L, the last. To correspond to this "higher order" exposition treatment, of course, a discovery treatment would have to be constructed so as to guide the learner into discovering some equivalent of $[(A + L)/2]N$.

† One way to get closer to a "more desired" sort of learning may be by imposing additional performance restrictions. See the next section.

In many other cases, of course, it may be impossible to identify an appropriate derivation principle due to the complexity of the situation. Under these circumstances learning by discovery would be indicated.

Insofar as performance capabilities are concerned, this analysis has not, as yet, been tested. The empirical results obtained in the "real world" may, as they often do, indicate inadequacies in the analysis. William Roughead and I have an experiment underway which hopefully will help provide further clarification.

Summary and Concluding Remarks

Let me first summarize briefly the basic characteristics of the set-function language (SFL). The principle, rather than the association, is assumed to be the basic behavior unit. Principles were defined as units of acquired knowledge which make it possible to give the appropriate response, in a class of responses, to each stimulus in a class of stimuli. In effect, the denotation of a *principle* (a function) is a set of ordered S-R pairs such that to each S exactly one R is assigned. If the responses in the denotative set are all identical, then the resulting principle is commonly called a *concept*. If only one S-R pair is involved, an *association* results.

The essential nature of any principle (i.e., internalized element of knowledge) can be characterized by an ordered four-tuple (I, D, O, R) where I refers to that set of stimulus properties (or dimensions) indicating when the principle is to be applied, D refers to the set of stimulus properties (or dimensions) from which the response dimension, R, is derived, and O is the combining operation or rule by which R is derived from the properties in D. In the case of the association, the properties in I and D are not only identical but may be viewed as simply referring to the stimulus itself. In concept learning, the corresponding properties refer to classes of stimuli. In principle learning, the properties refer to *dimensions* having values which are properties of a class of stimuli (e.g., X is a dimension referring to the numbers 2, 5, . . .).

To emphasize the distinction between the four elements characterizing a principle and symbolic representations of these elements, the latter are represented by primes. Any principle can be stated in the form, "If I', then $O'(D') \equiv R'$."

Let me reemphasize why I feel that some such language is needed. Much of the research in mathematics education, to date, has been of a fragmentary nature. A fundamental reason for this situation is that the wrong variables have too often been studied. By selecting easy to specify phenotypic variables, theory development has been made unnecessarily difficult. Although chronological age, for example, has been demonstrated to relate to a wide variety of behavioral phenomena, it is *not* an underlying cause. Time does not *cause* anything but only provides an occasion for things to happen. Age may, however, covary with underlying causes—e.g., the prerequisites for a given bit of learning. Integrative theories, leading to a more profound understanding of mathematics learning and teaching, can only be hoped for if more emphasis is given to the study of such underlying (genotypic) variables. Mathematics learning depends so critically on what the learner already knows and can do, what information is presented, and the test problems proposed that it seems unlikely that any profound understanding will eventuate from research which does not take into account the relationships involved. This will, of course, include a careful analysis of the mathematics itself, but cannot be limited to it.

A precise language, like the SFL, may not only help to guide the researcher in his quest for new and hopefully important questions but may be of even more help in formulating his questions in researchable form. Such a language may also help guard against the inappropriate generalization of research findings. One of the most important limitations of educational research has been that so little attention has been paid to determining the boundary conditions under which certain results will and will not obtain. Equally important, the SFL helps fill the gap between the highly controlled studies of the learning laboratory and the more encompassing but less well specified research of the mathematics educator. If nothing else, it may set the psychologist to wondering whether he has been studying the right things and the mathematics educator to searching for better and more precise means for coming to grips with his research problems. In particular, it is hoped that some guidelines will have been provided for the relatively inexperienced investigator and some food for thought for the active research worker in *psycho-mathematics*.

In spite of its relatively rigorous foundation the SFL has been used sparingly in the analyses and research described above. In relatively few cases was the language used formally. Such sparing use, however, is not unusual for scientific languages. Chemists, for example, refer to structural models only when needed to clarify difficult points; experimental psychologists use the S-R symbolism in a similar fashion. The important point is that by having a precise language to resort to when needed, fewer misunderstandings and more continuity of effort are to be expected.

Needed Research

Do I think that the SFL, in its present form, is sufficient to deal with most mathematics learning? Hardly! In the first place, many of the analyses proposed have not been put to experimental test. As we have seen in the study on rule generality a seemingly logical analysis of the situation is not necessarily fully reflected in the learner's performance. Psychology, in effect, is not logic and logic is not psychology. What frequently happens is that an analysis points up new questions, these questions are put to behavioral test, and the results indicate discrepancies in need of further analysis. This is the method of behavioral, nay any, science.

More important, in my quest for depth, I have had to sacrifice breadth. Let me list what I see as some of the most crucially needed research. First, no attempt has yet been made to extend this approach to deal with instruction which simultaneously involves several objectives.[25] Increasing the number of behavioral objectives imposes additional constraints on the nature of the learning to be achieved. The acquired knowledge must make it possible to perform successfully on a variety of tasks.

Second, complex problem solving and proving theorems, in which there are several stages, have not been considered. Examples have only been given for those cases in which the combining rule, O, was effectively an algorithm. Nothing was said about heuristics, those procedures which *might*, but will not necessarily, lead to a solution. As the reader will no doubt have noticed, even the earlier references to constructing models of finite geometries was reduced to a rather benign one-step principle.

Third, the question of whether the SFL can be used to deal with highly complex forms of interrelated knowledge is still open. A good deal of mathematical performance is apparently based on the acquisition of internalized models or images. On the basis of such knowledge, the learner can frequently respond appropriately to questions concerned with a whole class of principles rather than simply one. The problem is further compounded since, as

Easely[26] has argued so pervasively, there are important relationships between the various levels of mathematics learning: (*1*) concrete models, (*2*) the mathematical theory, itself, and (*3*) a suitable set of rules governing the syntax of the mathematical theory (i.e., a suitable logic). Can such learning be represented in the SFL and, if so, how?

Fourth, the SFL says little or nothing about efficiency or time to learn. Here, as was suggested in the last section, is where learning theories must play an important role.[27]

These are just four of the critical problems that need to be solved if a precise theory of mathematical learning is to be invented.* How they are to be solved, of course, is not entirely clear. Perhaps the procedures of task analysis[28] may be extended to deal with multiple objectives. Composite principles (i.e., functions) could well prove useful in dealing with problem solving and proving theorems, the component principles being used to represent individual steps in multistage processes where the effective stimulus is continually changing. Even heuristics may, in fact, turn out to be nothing more than sequences of algorithms.

Whatever the truth may be, we will never know unless many more mathematics educators are attracted into the prenatal field of *psycho-mathematics*.

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* To be sure, theories are invented, not discovered. Only facts are discovered.

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